Scott Topology and its Relation to the Alexandroff Topology

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WLOG: Without Loss of Generality, (57).

$\tau(\leq)$: T_0-Alexandroff space, (23).

$\sigma(P)$: Scott topology on the poset P, (35).

M : the set of all maximal elements of a poset X, (4).

m : the set of all minimal elements of a poset X, (4).

$M(A)$: the set of all maximal elements of a subset A of X, (4).

$m(A)$: the set of all minimal elements of a subset A of X, (4).

$\top$: top element in a poset, (4).

$\perp$: bottom element in a poset, (4).

$\uparrow x$: the set of all elements greater than or equal to x, (5).

$\downarrow x$: the set of all elements less than or equal to x, (5).

$\hat{x} := \uparrow x \cap M$, (29).

K_D: the set of all compact elements of a poset D, (8).

$\downarrow_K x$: the set of all compact elements less than or equal to x, (9).

$\sup D$ (or $\bigvee D$): the supremum(or the join) of the set D, (4).

$\inf D$ (or $\bigwedge D$): the infimum (or the meet) of the set D, (4).

$x \parallel y$: x is incomparable with y, (11).

$x \ll y$: x approximates y (or x is way-below y), (44).

$x \not\ll y$: x doesn't approximate y (or x is not way-below y), (46).

$\Downarrow x$: $\{y : y \ll x\}$; i.e., the set of all elements that approximate x, (45).

$\Uparrow x$: $\{y : \ x \ll y\}$; i.e., the set of all elements that x approximates, (45).

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Abstract

In this thesis, we survey the general topological concepts for the Scott topology, one of the fundamental foundations of theoretical computer science. We shall concentrate on the definition of the T_0-Alexandroff space and some of its topological identifications so that the relation between the Scott topology and the T_0-Alexandroff topology might be clearly discussed. We introduce here the property of being a T_0-space for the Scott topology and due to this we establish the main result for this research that the Scott topology and the T_0-Alexandroff topology coincide on finite posets while in general- every Scott open subset is open in T_0-Alexandroff topology and the converse need not be true.

The Main results of this research:

* In finite posets, each subset has a top element if and only if it is directed.

* In finite posets, every ideal is Scott-closed.

* On any poset X, the Scott topology is a T_0-space.

* A subspace of Scott topology is a Scott subspace.

* Every algebraic dcpo is continuous.

* In a continuous finite poset P, no proper subset is a basis for P.

Introduction

Topology is thought of as one of the branches of mathematics that has applications in practical life. Nowadays, topology has proved to be an essential tool for certain aspects of theoretical computer science [13]. Conversely, the problems that arise in the computational setting have provided new and interesting stimuli for topology [13]. These problems also have increased the interaction between topology and related areas of mathematics such as order theory and topological algebra [13] [Order theory is a branch of mathematics that studies various kinds of binary relations that capture the intuitive notion of ordering, providing a framework for saying when one thing is "less than" another. Domain theory deals with partially ordered sets to model a domain of computation. The goal is to interpret the elements of such an order as pieces of information or (partial) results of a computation, where elements that are higher in the order extend the information of the elements below them in a consistent way[23]].

A concept that plays an important role in the theory is the one of a directed subset of a domain, i.e. of a non-empty subset of the order in which each two elements have some upper bound [23]]. In view of our intuition about domains, this means that every two pieces of information within the directed subset are consistently extended by some other element in the subset. Hence we can view directed sets as consistent specifications, i.e. as sets of partial results in which no two elements are contradictory [23].

In fact, the concept of the directed set paves to the fundamental concept in this research namely, the directed-complete poset (in short, dcpo) which occupies a large area in this research.

In this thesis, we survey the general topological concepts for the Scott topology -which is of fundamental importance in domain theory since it lies at the heart of the structure of domains([12])-. Also, this research compares between the Scott topology and one of the Alexandroff topology types; namely the T_0-Alexandroff topology.

This thesis consists of four chapters: the Preliminaries, the Alexandroff space, the Scott topology and finally the Scott topology and approximation relation.

In the first chapter - which is divided into three sections-, we begin the first section with an essential definition; the definition of the partial order relation and consequently the definition of a poset . Then, we pave the way to represent, diagrammatically, the elements of the poset according to its partial order. The definitions of some terminologies related to the whole set or to the position of the elements within it are given, such as: maximal, maximum, minimal, infimum, up set (or upper set as some authors prefer) or down set . . . etc.

In the second section, we give the definition of the directed-complete poset (briefly, dcpo) which is of special importance in the last two chapters. Also, we introduce the definition of the algebraic dcpo. In the third section,we introduce some related topological concepts; since this is a topological issue and the topological concepts will naturally be used.

The second chapter consists of three sections. The first paves the way to the Alexandroff topology by a quick historical introduction including the definition of the Alexandroff topology as a special class of topologies and then its specialization order.

The second section goes more closer and gives the definitions of the open sets and closed sets in the T_0-Alexandroff space. The third section of this chapter gives the concepts of the interior, closure, the boundary and the derived set of a subset of this space.

The third chapter-which is the main one- is divided into two sections. The first, gives the definition of the Scott-open set and then shows that the collection of all Scott open sets forms a topology called the Scott topology. Also, it shows that the Scott topology is sober over an algebraic dcpo. The base of the Scott topology is given by means of the set of all compact elements. The second compares between the Scott topology and the Alexandroff topology on finite sets and in general.

The last chapter with its three sections gives the definitions of an important concepts. The first section introduces the concept of the approximation (or the way-below) relation and some of its fundamental properties. By the use of this essential concept, the definition of the continuous poset comes in the second section. Finally, the base of the Scott topology on a poset P is defined on the collection of the approximation relation of each element in the poset P. Also the base of the Scott topology is defined by the use of filters.

At the end of this introduction, we would like to say that collecting the items of this subject wasn't so easy since we have had no paper in our hands talking explicitly about the Scott topology. What we have found was some definitions and propositions. Most of what has been found was without proof, and if there was any, it was in short and need to be reproved again. So, most of the proofs in this research has been written by us. Not only this but also the rearrangements of the research; naming the chapters and the sections together with the comments in the beginning of the chapters and the sections and between lines. It was not so easy to accomplish all these things without the help of the supervisor.

Chapter 1

Preliminaries

The interaction between topology and order theory has a plenty soil over computer science. Domain theory, where this reaction happens, deals with a special relation over a set P and this relation orders the elements of P. This relation is called a *partial order* and this together with the set under consideration is called a *poset*. These definitions and more will be given in the current chapter together with some related topological concepts in order to be ready to study the next chapters.

1.1 Partially Ordered Sets

Definition 1.1.1. [18] A relation $\leq$ on a set P is called *partial order* (simply *order*) on P if for every $a, b, c \in P$:
(i) $a \leq a$ (reflexivity),
(ii) $a \leq b$ and $b \leq a$ implies $a = b$ (anti-symmetry),
(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).
The set P together with a partial order $\leq$ is called a *partially ordered set* (briefly a *poset*).

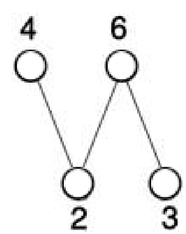
Example 1.1.1. *The set* $\mathbb{N}$ *of all natural numbers forms a poset under the usual order* $\leq$ *on* $\mathbb{R}$. *Similarly, the set of integers* $\mathbb{Z}$, *rationales* $\mathbb{Q}$ *and real numbers* $\mathbb{R}$ *under the usual order* $\leq$ *form posets.*

Example 1.1.2. *Let* X *be a set. The set* $\mathcal{P}(X)$ *of all subsets of* X *under the relation "contained in"* ($\subseteq$) *forms a poset.*

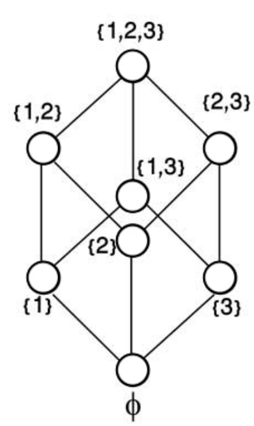
Diagrammatical representation of a poset:

Each poset can be represented by the help of a diagram. To draw the diagram of a poset, we represent each element by a small circle (a dot) and any two comparable elements are joined by lines in such a way that if $a \leq b$ then a lies below b in the diagram. Non-comparable elements are not joined. Thus, there will not be any horizontal lines in the diagram of a poset.

Example 1.1.3. *The set* $\{2, 3, 4, 6\}$ *under divisibility relation forms a poset with a diagram as below:*

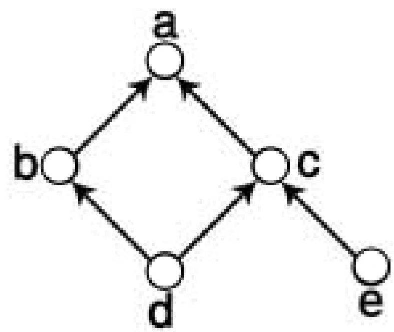


Example 1.1.4. *If* $X = \{1, 2, 3\}$, *then the poset* $\mathcal{P}(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ *under* $\subseteq$ *relation is represented by the diagram below.*



Example 1.1.5. *Let $X = \{a, b, c, d, e\}$. Then, the diagram below defines a partial order on X as follows :*

$x \leq y$ if and only if $x = y$ or one can go from x to y in the diagram in the indicated direction; i.e., upward.



Definitions 1.1.2. [18]and [5] 1. Let a, b be two elements in an ordered set. We say that a *precedes* b (or a is *smaller* than b) if $a \leq b$. In this case we say that b *follows* a or *larger* than a. Furthermore, we write $a < b$ if $a \leq b$ and $a \neq b$.

2. A subset C of a poset P is a *chain* if any two elements of C are comparable. Alternative names for a chain are *linearly ordered set* and *totally ordered set*. Thus, if C is a chain and $x, y \in C$ then either $x \leq y$ or $y \leq x$.

3. If a relation R on a set A (which is a subset R of $A \times A$) defines a partial order, then the inverse relation R^{-1} is also a partial order; it is called the *inverse order* (or the *dual order*).

Definition 1.1.3. [18] Let A be a subset of a poset X. Then, the order in X induces an order in A in the following natural way: If $a, b \in A$, then $a \leq b$ as elements in A if and only if $a \leq b$ as elements in X.

Equivalently, if R is a partial order in X, then the relation $R_A = R \cap (A \times A)$ - called the *restriction* of R to A - is a partial order in A. The ordered set(A, R_A) is called *partially ordered subset* of the ordered set (X, R).

It should be noted that a chain C as an ordered subset of X is totally ordered. Clearly, if X is totally ordered, then every subset of X will be totally ordered.

Definition 1.1.4. [18] Let X be an ordered set. An element $a \in X$ is called *maximal* if whenever $a \leq x$ then $x = a$, that is; if no element in X follows a except a. Similarly, an element $a \in X$ is called *minimal* if whenever $x \leq a$ then $x = a$, that is; if no element in X precedes a except a itself.

We denote the set of all maximal (resp. minimal) elements of an ordered set X by M (resp. m). If A is any subset of X, we write $M(A)$ (resp. $m(A)$) to denote the set of maximal (resp. minimal) elements of A under the induced order.

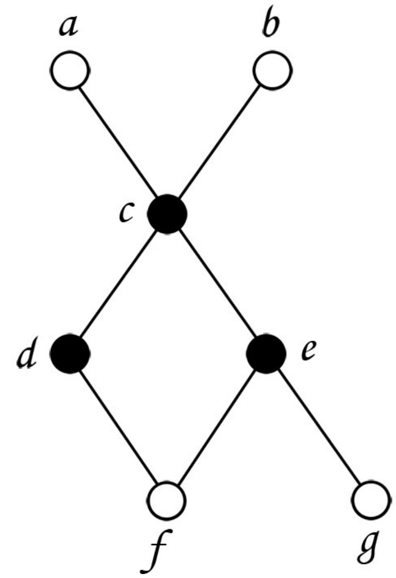
If there is an element $\top \in X$ such that $x \leq \top$ for all $x \in X$, then $\top$ is called *maximum* (or *top*) element. On the other hand, if there is an element $\bot \in X$ such that $\bot \leq x$ for all $x \in X$, then $\bot$ is called *minimum* (or *bottom*) element.

It should be noted that the set M of all maximal elements of a poset X may be an empty set. In the case where $|M| = 1$ (M contains only one element), the set X has a top element $\top$. Dually, if $|m| = 1$, then X has a bottom element $\bot$.

Example 1.1.6. *Recall Example 1.1.5. The element "a" is maximal while both d and e are minimal elements.*

Definitions 1.1.5. [18] Let A be a subset of a poset X. An element $u \in X$ is an *upper bound* of A if $x \leq u \ \ \forall\, x \in A$. The *least upper bound* (or the *supremum*) of A - denoted by $\sup A$ (or $\bigvee A$)- is an upper bound that precedes each upper bound of A. An element $\ell \in X$ is a *lower bound* of A if $\ell \leq x \ \ \forall\, x \in A$. The *greatest lower bound* (or the *infimum*) of A - denoted by $\inf A$ (or $\bigwedge A$)- is a lower bound of A that follows each lower bound of A. For a subset A, $\sup A$ and $\inf A$ may not exist. A is said to be *bounded above* if it has an upper bound, and *bounded below* if it has a lower bound. If A has both upper and lower bounds, then it is *bounded*. For a subset A of a poset X, the set of all upper (resp. lower) bounds of A is denoted by A^u (resp. A^ℓ).

Example 1.1.7. *[18] Let $X = \{a, b, c, d, e, f, g\}$ be a set ordered by the following diagram:*



Let $B = \{c, d, e\}$. The elements a, b and c are upper bounds of B and f is the only lower bound of B. The element g is not a lower bound of B, since g doesn't precede d. Moreover, $\inf B = f \notin B$, *while* $\sup B = c \in B$.

Definition 1.1.6. [5] We say y covers x or x is *covered* by y and write $y \leftarrowtail x$ or $x \rightarrowtail y$ if $x < y$ and when $x \leq z < y$ then $z = x$.

Definition 1.1.7. [5] A subset O of a poset P is a *down set* (or *lower set*) if, whenever $x \in O$ and $y \leq x$, then we have $y \in O$. On the other hand, a subset U of a poset P is an *up set* (or *upper set*) if, whenever $x \in U$ and $x \leq y$, we have $y \in U$. For $x \in P$, we define the down set $\downarrow x = \{y \in P : \ y \leq x\}$; and the up set $\uparrow x = \{y \in P : x \leq y\}$. For a set $B \subseteq P$, we define the down set $\downarrow B = \{y \in P : (\exists x \in B)\ y \leq x\}$ and the up set $\uparrow B = \{y \in P : (\exists\ x \in B)\ x \leq y\}$. In this case, $\uparrow x = \uparrow \{x\}$ and $\downarrow x = \downarrow \{x\}$.

If A is a down set of P, then the complement A^c is an up set, since if $a \in A^c$ and $a \leq b$, then $b \in A^c$ by a contrapositive argument.

Example 1.1.8. *Let $\mathbb{R}$ be the set of all real numbers with its usual order. Let $A, B \subseteq \mathbb{R}$ be such that $A = [3, \infty)$ and $B = (-\infty, 0]$, then A is an up set and B is a down set.*

Definition 1.1.8. [5] A poset P satisfies the *ascending chain condition* (ACC) , if for any increasing sequence $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ in P, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \ldots$. The dual of (ACC) is the *descending chain condition* (DCC). If a poset satisfies both ACC and DCC, we say P is of *finite chain condition* (FCC).

Example 1.1.9. *A collection of subsets of a finite set X when ordered by inclusion satisfies the ACC and if ordered by reverse inclusion it satisfies the DCC.*

Example 1.1.10. *Each finite poset is of FCC.*

1.2 Directed - complete posets (dcpo)

A special kind of posets is considered in this section, and its importance will appear later. Here is the definition of this kind:

Definition 1.2.1. [4] Let $(D, \leq)$ be a partially ordered set. A subset U of D is called *directed* if U is inhabited (i.e., $U \neq \phi$) and $\forall\ u, v \in U$, $\exists\ w \in U$ such that $u \leq w$ and $v \leq w$.

Example 1.2.1. *Recall Example 1.1.2 for a non-empty set X, we have that $(\mathcal{P}(X), \subseteq)$ is a directed set, since for any non-empty subsets $A, B \in \mathcal{P}(X)$, take $C = A \cup B$ for the second condition.*

Lemma 1.2.2. *[10] Let P be a poset. A non-empty chain in P is directed.*

Proof. Let P be a poset and let U be a non-empty chain in P. Let $u, v \in U$. Since in the chain, each two elements are comparable, then $u \leq v$ or $v \leq u$. If $u \leq v$, then $u \leq v$ and $v \leq v$. Similarly, if $v \leq u$, we have $v \leq u$ and $u \leq u$. Thus, U is directed. □

Example 1.2.2. *The set of natural numbers* $\mathbb{N}$*, the set of integers* $\mathbb{Z}$*, the set of rationales* $\mathbb{Q}$ *and the set of real numbers* $\mathbb{R}$ *are directed sets under the usual order.*

Lemma 1.2.3. *Let* P *be a poset. Then, for any* $x \in P$*, the set*

$$\downarrow x = \{y \in P : y \leq x\}$$

is directed with x *as its join.*

Proof. For any $y, z \in \downarrow x$, we have $y \leq x$ and $z \leq x$. Thus $\downarrow x$ is directed. Since $y \leq x$ for each $y \in \downarrow x$, then x is an upper bound in $\downarrow x$, and hence $x = \bigvee \downarrow x$. □

Proposition 1.2.4. *In a finite poset* P*, a subset has a top element "* $\top$ *" if and only if it is directed.*

Proof. Let P be a finite poset.

($\Rightarrow$) Let $U \subseteq P$ be a non-empty subset with a top element $\top_U$. Then for any $u \in U$, $u \leq \top_U$. Consequently, $\forall\ u, v \in U$, take $w = \top_U \in U$ so that $u \leq w$ and $v \leq w$. Thus, U is directed.

($\Leftarrow$) Let $U \subseteq P$ be a directed subset. Then, $U \neq \phi$. Since P is finite, then so is U. Let $U = \{u_1, u_2, \ldots, u_n\}$. Now, for any $u_i, u_j \in U$, $\exists\ u_k \in U$ such that $u_i \leq u_k$ and $u_j \leq u_k$. Also, for any $u_m \in U$, $\exists\ u_w \in U$ such that $u_k \leq u_w$ and $u_m \leq u_w$. Thus, by the transitivity of " $\leq$ " and the directedness of U we have $u_w = \max\{u_i, u_j, u_k, u_m, u_w\}$. Continuing in this fashion our process must come to an end since U is finite. That is; there must be an element $u \in U$ such that $u = max\{u_1, u_2, \ldots, u_n\}$. Hence, U has a top element. □

Definitions 1.2.5. [4] (1) A partially ordered set D is called *directed-complete* (briefly dcpo) if every directed subset has supremum.

(2) A *complete partially order set* (briefly cpo) is a dcpo with a least element.

Example 1.2.3. *[21] Every finite poset is a dcpo.*

Proof. By Proposition 1.2.4, in finite posets, each directed subset has a top element and hence, has a supremum. Thus, we are done. □

Example 1.2.4. *The set of real numbers $\mathbb{R}$, the set of rationales $\mathbb{Q}$, the set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$ fail to be dcpos under the usual order, since all these sets are directed subsets of themselves and no one has a supremum. Moreover, each finite subset of any of them is a dcpo under the usual order. Furthermore, each finite subset is a cpo.*

The following definition plays a big rule in the later chapters.

Definition 1.2.6. [4] An element a of a dcpo D is called *compact* if, for any directed subset U of D, $a \leq \bigvee U$ implies that $\exists u \in U$ such that $a \leq u$ (i.e.,$\uparrow a \cap U \neq \phi$). The set of all compact elements of D will be denoted by K_D

Lemma 1.2.7. *[4] Whenever it exists, the supremum of any finite set of compact elements is compact.*

Proof. Let D be a dcpo and let $A = \{a_i\}_{i=1}^{n}$ be a finite set of compact elements in D. Suppose that the supremum $b \in D$ of A exists. By the definition of the supremum, we have $a_i \leq b \, \forall \, a_i \ \in A$. Now, let U be any directed subset of D such that $b \leq \bigvee U$. So, we have $a_i \leq b \leq \bigvee U \ \ \forall \, a_i \in A$. Since a_i is compact $\forall \, i$, there exists $u_i \in U$ such that $a_i \leq u_i$ for all $i = 1, 2, 3, \ldots, n$. Let $u = \max\{u_1, u_2, \ldots, u_n \ \}$. Then, u exists in U by the argument of Proposition 1.2.4. Thus, $a_i \leq u \ \ \forall \, a_i$ in A. Hence u is an upper bound of A. Since b is the least upper bound of A, then $b \leq u$ and so b is compact. □

Proposition 1.2.8. *In a dcpo,D, if each directed subset of D contains its supremum, then $K_D = D$.*

Proof. Clearly $K_D \subseteq D$. Now, let $a \in D$ and let U be a directed subset of D such that $a \leq \bigvee U$. Since $\bigvee U \in U$ (by hypothesis), then take $u = \bigvee U \in U$ and so $a \leq u$. Therefor, a is compact. □

Corollary 1.2.9. *For each finite subset B of $\mathbb{N}$ (or $\mathbb{Z}$), $K_B = B$.*

Proof. By Example 1.2.3, B is a dcpo. Since any finite subset of $\mathbb{N}$ (or $\mathbb{Z}$) contains its supremum, then from Proposition 1.2.8, $B = K_B$. $\square$

Proposition 1.2.10. *Let P be a poset that satisfies ACC and let $A \subseteq P$. Then, A is directed if and only if $\mid M(A) \mid =1$. That is; A is directed if and only if A has a supremum.*

Proof. Let A be a non-empty subset of a poset P where P satisfies the ACC.
($\Rightarrow$) Let $x_1, x_2 \in M(A) \subseteq A$. So, there exists $u \in A$ such that $x_1 \leq u$ and $x_2 \leq u$. But x_1, x_2 are maximal in A. So $x_1 = u$ and $x_2 = u$. Therefore, $x_1 = x_1$.
($\Leftarrow$) Suppose that $\mid M(A) \mid = 1$. So, $\bigvee A$ exists in A. Therefore, $\forall\ x, y \in A,\ x \leq z$ and $y \leq z$, where $z = \bigvee A$. Hence, A is directed. $\square$

Definition 1.2.11. [4] A dcpo D is called *algebraic* if, for every $x \in D$, the set $\downarrow_K x = \{a \in K_D : a \leq x\}$ is directed and $x = \bigvee \downarrow_K x$. Alternately, algebraic dcpo's are referred to as *domains*(see [6]).

Lemma 1.2.12. *In an algebraic dcpo D, the set $\downarrow_K x$ is non-empty for each $x \in D$.*

Proof. Since D is an algebraic dcpo, then for each $x \in D$, the set $\downarrow_K x$ is directed and hence $\downarrow_K x \neq \phi$. $\square$

Example 1.2.5. *Let $P = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$, with natural order. Then, P is an algebraic dcpo.*

Proof. Clearly, P is a finite poset and hence is a dcpo. Also, $P = K_P$ (see Corollary 1.2.9). Therefore, for any $x \in P$, $\downarrow_K x = \downarrow x = \{m \in \mathbb{N} : m \leq x\}$ which is directed and has x as its join (see Lemma 1.2.3). Hence, P is an algebraic dcpo. $\square$

Example 1.2.6. *Let $D = (-\infty, 1]$ with the usual order. It is clear that D is a dcpo. Moreover, if $y \in D$ and for $x \leq y$, let U be the interval of real numbers (x, y), which is a directed subset of D with $y = \bigvee U$. So, $y \leq \bigvee U = y$ and there is no $u \in U$*

such that $y \leq u$. Therefore, y is not compact and consequently, $K_D = \phi$. Hence, $D$is not algebraic.

The following example shows a non-algebraic dcpo with non-empty set of compact elements.

Example 1.2.7. *Let $P = [0,1]$ with usual order. Then, P is a dcpo. Since 0 is the bottom element in P, then for any directed subset U of P with $0 \leq \bigvee U$, there is $u \in U$ such that $0 \leq u$ ($U \neq \phi$ since it is directed). Thus, $0 \in K_P$.*
Now, for any $x \in P$ with $x \neq 0$, we have $U = (z, x), z \in P$ is directed with $\bigvee U = x$ but U contains no element u such that $x \leq u$. Thus, $x \notin K_P$. Hence, $K_P = \{0\}$ and so $\forall\ x \in P$, $\downarrow_K x = \{0\}$ which is directed with 0 as its join ; i.e., x is not its join except for $x = 0$. Thus, P is not algebraic.

Corollary 1.2.13. *If D is a finite algebraic dcpo, then each element in D is compact. That is; $D = K_D$.*

Proof. Let D be a finite algebraic dcpo and let $x \in D$. Then, $\downarrow_K x$ is a finite directed subset of compact elements with x as its join. Thus, by Lemma 1.2.7, x is compact. $\square$

As a generalization of Example 1.2.5, we have the following lemma:

Lemma 1.2.14. *Any finite linearly ordered set is an algebraic dcpo.*

Proof. Let P be a finite linearly ordered set. Then P is a dcpo and $P = K_P$. Thus, for any $x \in P$, $\downarrow_K x = \downarrow x$ which is a directed set with x as its join. Hence, P is an algebraic dcpo. $\square$

In any algebraic dcpo D, the ordering relation between its elements can be recovered from the structure of K_D.

Proposition 1.2.15. *[4] Let D be an algebraic dcpo. For $x, y \in D$, $x \leq y$ if and only if $\downarrow_K x \subseteq \downarrow_K y$, that is; $x \leq y$ if and only if $\forall\, a \in K_D$, $a \leq x$ implies $a \leq y$.*

Proof. ($\Rightarrow$) Suppose that $x \leq y$ and let $a \in \downarrow_K x$. Then, $a \leq x$. Since $x \leq y$, then $a \leq y$, that is; $a \in \downarrow_K y$. Thus, $\downarrow_K x \subseteq \downarrow_K y$.
($\Leftarrow$) Suppose that $\downarrow_K x \subseteq \downarrow_K y$. Then $\bigvee \downarrow_K x \leq \bigvee \downarrow_K y$. Since D is algebraic, so $\bigvee \downarrow_K x = x$ and $\bigvee \downarrow_K y = y$ and hence, $x \leq y$. □

Before we close this section let us pass on to the dual of the notion " directed set", namely the " filtered set".

Definition 1.2.16. [7] A non-empty subset S of a poset P is said to be *filtered* if given $x, y \in S$, there exists $z \in S$ such that $z \leq x$ and $z \leq y$.

Example 1.2.8. *Any subset of $\mathbb{N}$ ($\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$) is a filtered set under the usual order. In general, each chain is a filtered set.*

Lemma 1.2.17. *If P is any poset with bottom element $\perp$, then any subset of P containing $\perp$ is a filtered set.*

Proof. Let P be a poset with bottom element $\perp$ and let S be any subset of P such that $\perp \in S$. Then, for any $x, y \in S$, $\perp \leq x$ and $\perp \leq y$. Hence, S is a filtered set. □

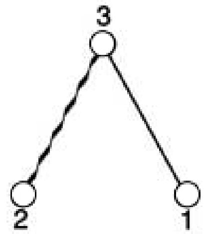
Corollary 1.2.18. *Every cpo is a filtered set.*

Proof. Straightforward (See Definition 1.2.5 part (2)). □

Definition 1.2.19. [7] Let P be a poset. A non-empty subset F of P is called a *filter* if F is a filtered up set.

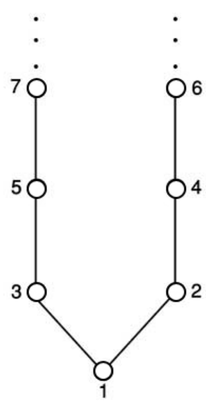
Example 1.2.9. *Let the poset $D = [2, 3] \cup \{1\}$ and suppose that the elements of D are ordered as follows: the elements of the closed interval $[2, 3]$ are ordered by the usual order " $\leq$". For 1 and any $x \in [2, 3)$ we have $x \| 1$ (i.e. x and 1 are* *incomparable). Finally, if $x = 3$ then $1 \leq 3$.*

We will use the broken line to represent the usual order for all the entries of the interval, while the line segment to represent the order between two elements only.



Now, let $S = [a, 3] \subseteq [2, 3]$, where $a < 3$. Clearly S is a filter. The set $U = [2, 2.5] \cup \{3\}$ is a filtered set which isn't a filter (since it isn't an up set). Consider the set $A \subseteq D - \{3\}$. Then $1 \in A$ and A contains more than one point. Moreover, A is not a filtered set, since for any $x \in A - \{1\}$, there is no element z in A such that $z \leq x$ and $z \leq 1$.

Example 1.2.10. *Let $P = \mathbb{N}$, the set of all natural numbers, be ordered as follows: $1 \leq 2 \leq 4 \leq 6 \leq \ldots$ and $1 \leq 3 \leq 5 \leq 7 \leq \ldots$, and $x \parallel y$ $\forall$ $x \in \{2, 4, 6, \ldots\}$ and $y \in \{3, 5, 7, \ldots\}$. Clearly P is a filter.*

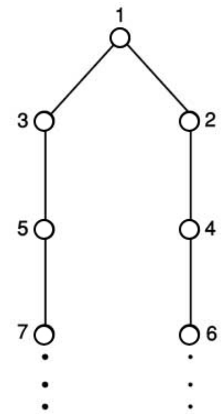


Definition 1.2.20. [7] A non-empty subset I of a partially ordered set $(P, \leq)$ is an *ideal*, if the following conditions hold:

(i) I is a down set. That is; for every $x \in I$, $y \leq x$ implies that y is in I.

(ii) I is a directed set. That is; for every $x, y \in I$, there is some element $z \in I$, such that $x \leq z$ and $y \leq z$.

Example 1.2.11. *Let $P = \mathbb{N}$, the set of all natural numbers, be ordered by the reverse order of that in Example 1.2.10. Clearly P is an ideal.*



1.3 Topological concepts

Many topological concepts will be used in the next chapters, so we give here their definitions to keep the next chapters for the discussion of the new and related concepts.

We are going to introduce first a structure on a set X by use of a collection of subsets of X. For this, we shall soon set forth a set of axioms which a collection of subsets must obey in order to fall within the circle of our studies. Any collection of subsets of X satisfying these axioms will be called a *topology* of X[15].

Definition 1.3.1. [15] Let $X \neq \phi$ be a set. Then a *topology* on X is a subset τ of $\mathcal{P}(X)$(the power set of X) obeying the following axioms:
(a) X and ϕ belong to τ.
(b) If U_1 and U_2 belongs to τ, then $U_1 \cap U_2$ belongs to τ.
(c) If $\{U_\alpha : \alpha \in \Delta\}$ is an indexed family of sets, each of which belong to τ, then $\bigcup\limits_{\alpha \in \Delta} U_\alpha$ belongs to τ.

We shall call the elements of a topology on any set X, *open* subsets of X.

Definition 1.3.2. [15] A *topological space* is a set X together with a topology τ on X. The notion (X, τ) will often be used for a topological space, but the shortened notation *the space* X will also be used when no confusion arises concerning the topology on X.

When $\mathbb{R}$ is the set under consideration, the standard topology will always be assumed, unless otherwise explicitly stated.

Definition 1.3.3. [19] Given two topologies τ_1 and τ_2 on a set X. We say that τ_1 is *weaker (smaller,coarser)* than τ_2, or τ_2 is *stronger(larger,finer)* than τ_1 if $\tau_1 \subseteq \tau_2$.

Example 1.3.1. *The left ray topology on* $\mathbb{R}$ *is coarser than the usual topology, since each set of the form* $(-\infty, a)$ *which is open in the left ray topology, is also open in the usual topology while the set* $A = (1, 9)$ *belongs to the usual topology but not to the left ray topology.*

Definition 1.3.4. [19] If X is a topological space and $E \subseteq X$, we say E is *closed* if $X - E$ is open.

Theorem 1.3.5. *[19] If* $\mathcal{F}$ *is the collection of closed sets in a topological space* X*, then*
(a) X *and* ϕ *both belong to* $\mathcal{F}$*,*
(b) any intersection of members of $\mathcal{F}$ *belongs to* $\mathcal{F}$*,*
(c) any finite union of members of $\mathcal{F}$ *belongs to* $\mathcal{F}$*.*

Definition 1.3.6. [19] If X is a topological space and $E \subseteq X$, the *closure* of E in X is the set

$$\overline{E} = Cl(E) = \bigcap\{K : K \text{ is a closed set containing } E\}.$$

When confusion is possible as to what space the closure is to be taken in, we will write $Cl_X(E)$.

Remark 1.3.7. : By part (b) of Theorem 1.3.5, it is clear that $\overline{E}$ is closed and it is the smallest closed set containing E. The intersection of all closed sets containing E is closed, where the precise meaning of "smallest" is that if the sets containing E is ordered by $K_1 \leq K_2 \;\; iff \;\; K_1 \subseteq K_2$.

Theorem 1.3.8. *[15] A subset A of a space X is closed if and only if $\overline{A} = A$.*

Some properties of the closure are now considered:

Theorem 1.3.9. *[15] Let A and B be subsets of the space X. Then*
(a) $\overline{\phi} = \phi$.
(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) $\overline{\overline{A}} = \overline{A}$.
(d) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.

Definition 1.3.10. [15] Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is an *interior point* of A if there exists an open set U containing x such that $U \subseteq A$. The set of interior points of A is called the interior of A and is denoted by $\mathrm{Int}(A)$ (or A^o). A point $x \in X$ is an *exterior point* of A if there exists an open set U containing x such that $U \cap A = \phi$. The set of exterior points of A is called the exterior of A and is denoted by $\mathrm{Ext}(A)$ (or$(X - A)^o$).A point $x \in X$ is a *boundary point* of A if every open set in X containing x contains at least one point of A, and at least one point of $X - A$. The set of boundary points of A is called the boundary of A and is denoted by $\mathrm{Bd}(A)$ (or $Fr_X(A)$).

Definition 1.3.11. [15] Let X be a topological space, $x \in X$, and $A \subseteq X$. Then x is a *cluster point* (or *accumulation point*, *limit point*) of A if every open set containing x contains at least one point of A different from x. For any set A in the space X, the set of all cluster points of A is called the *derived set of* A. The derived set of A is denoted by A'.

We can write $x \in A'$ if and only if $\forall\, U \in \tau$ such that $x \in U,\ U \cap (A - \{x\}) \neq \phi$. The relation between the closure and the derived sets of a set A is introduced in the following theorem:

Theorem 1.3.12. *[19]* $\overline{A} = A \cup A'$.

Definition 1.3.13. [15] Let X be a topological space. Then, $A \subseteq X$ is *dense* in X if $\overline{A} = X$.

Definition 1.3.14. [18] A subset A of a topological space X is said to be *nowhere dense* in X if the interior of the closure of A is empty, i.e., $(\overline{A})^o = \phi$.

Definition 1.3.15. [15] Let (X, τ) be a topological space. A *base* for τ is a collection $\mathcal{B}$ of subsets of X such that:
(a) each member of $\mathcal{B}$ is also a member of τ and
(b) if $U \in \tau$ and $U \neq \phi$, then U is the union of sets belonging to $\mathcal{B}$.

Since $\mathcal{B} \subseteq \tau$, and using part (b) of the definition provided that $U \neq \phi$, we have that, if $U \neq \phi$, then $U \in \tau$ if and only if U is the union of members of $\mathcal{B}$. Therefore, a base for τ completely determines τ by arbitrary unions of members of $\mathcal{B}$. Also we see that any topology is a base for itself. So, any topology has at least one base.

Theorem 1.3.16. *[19] $\mathcal{B}$ is a base for a topology on X if and only if:*
(a) $X = \bigcup\limits_{B \in \mathcal{B}} B$ *and*
(b) whenever $B_1, B_2 \in \mathcal{B}$ with $p \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ with

$$p \in B_3 \subseteq B_1 \cap B_2.$$

Definition 1.3.17. [19] If X is a topological space and $x \in X$, a *neighborhood* (abbreviated *nhood*) of x is a set U which contains an open set V containing x. Thus, evidently, U is a nhood of x if and only if $x \in U^o$. The collection $\mathcal{U}_x$ of all nhoods of x is called the nhood system of x.

Definition 1.3.18. [19] A *nhood base* at x in the topological space X is a subcollection $\mathcal{B}_x$ taken from the nhood system $\mathcal{U}_x$ having the property that each $U \in \mathcal{U}_x$ contains some $V \in \mathcal{B}_x$.

Theorem 1.3.19. *[19] Let X be a topological space and suppose a nhood base has been fixed at each $x \in X$. Then,*

$$\overline{A} = \{x \in X : \textit{each basic nhood of } x \textit{ meets } A\}$$

Definition 1.3.20. [19] If (X, τ) is a topological space and $A \subseteq X$, the collection $\tau_A = \{G \cap A : G \in \tau\}$ is a topology on A, called the *relative topology* of A. This topological space is denoted by (A, τ_A).

The fact that a subset of X is being given this topology is signified by referring to it as a subspace of X.

When a topology is used on a subset of a topological space without explicitly being described, it is assumed to be the relative topology.

Definition 1.3.21. [15] Let(X, τ) be a topological space. Then,

(i) the space (X, τ) is a T_o-space if for each pair of distinct points $x, y \in X$, there is either an open set containing x but not y or an open set containing y but not x,

(ii) the space (X, τ) is a T_1-*space* if for each pair of distinct points $x, y \in X$, there is an open set in X containing x but not y and an open set in X containing y but not x ,

(iii) the space (X, τ) is called a T_2-*space* (or a *Hausdorff space*) if for each pair of distinct points $x, y \in X$, there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.

In fact, (i),(ii) and (iii) in the above definition are always given within a package known as *the Separation Axioms.*

Example 1.3.2. *The indiscrete topology on a set is not a T_o-space, while the discrete topology on a set is not only T_o but also T_1 and T_2 (singletons are open in the discrete topology).*

Remark 1.3.22. Every T_1-space is a T_0-space and every T_2-space is a T_1-space.

Definition 1.3.23. [15] Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of subsets of the space X and $B \subseteq X$. The family $\{A_\alpha : \alpha \in \Delta\}$ *covers* B if $B \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$. If Δ is finite and $\{A_\alpha : \alpha \in \Delta\}$ covers B, then $\{A_\alpha : \alpha \in \Delta\}$ is called a *finite cover* of B. If each A_α, $\alpha \in \Delta$, is open (closed) in X and $\{A_\alpha : \alpha \in \Delta\}$ covers B, then $\{A_\alpha : \alpha \in \Delta\}$ is called an *open (closed) cover* of B.

Definition 1.3.24. [15] Let $\{A_\alpha : \alpha \in \Delta\}$ be a cover of $B \subseteq X$. Then the family $\{A_\beta : \beta \in \Omega \subseteq \Delta\}$ is a *subcover* of $\{A_\alpha : \alpha \in \Delta\}$ of B if $\{A_\beta : \beta \in \Omega \subseteq \Delta\}$ is a cover of B.

Definition 1.3.25. [15] A space X is called *compact* if each open cover of X has a finite subcover. A subset A of the space (X, τ) is compact if the space (A, τ_A) is compact.
Indeed, to prove a space is not compact, we need only exhibit one open cover which has no finite subcover.

Definition 1.3.26. [18] A space X is *locally compact* if and only if every point in X has a compact nhood.

Proposition 1.3.27. *Every compact space is locally compact.*

Theorem 1.3.28. *[22] Let $\mathcal{B}$ be a base for a topological space X. Then, X is compact if and only if each cover $\{B_\alpha \in \mathcal{B} : \alpha \in \Delta\}$ of X can be reduced to a finite subcover.*

Chapter 2

Alexandroff Space

2.1 Introduction

In this section we study a class of topological spaces called T_o-Alexandroff spaces. An Alexandroff space is a topological space (X, τ) that satisfies the property that a finite intersection of open sets is open. What about an arbitrary intersection of open sets?. This property doesn't hold in all topological spaces. For example, in the standard topology on $\mathbb{R}$,

$$\bigcap_{n\in\mathbb{N}}(-\frac{1}{n}, \frac{1}{n}) = \{0\}$$

which is not open in $\mathbb{R}$.

In fact, this property holds in a special class of topological spaces called Alexandroff spaces. This subject was first studied in 1937 by P. Alexandroff [14] under the name of Diskrete Räume (discrete space). The name is not valid now, since a discrete space is a space where the singletons are open. He gave an example of a T_o-Alexandroff space on a poset$(P, \leq)$ taking $\mathcal{B} = \{\uparrow x : x \in P\}$ to be the unique minimal base. The induced topology on P -denoted by $\tau(\leq)$- is a T_o-Alexandroff space. If (X, τ)

is an Alexandroff space, he defined its *(Alexandroff) specialization order* $\leq_\tau$ on X as follows:

$$\forall\ a, b \in X, a \leq_\tau b\ if\ a \in \overline{\{b\}}. \tag{2.1.1}$$

The specialization order is reflexive and transitive. It is antisymmetric - and hence a partial order - if and only if X is T_o. Moreover, if $(X, \leq)$ is a poset and if $\tau(\leq)$ is its induced T_o-Alexandroff topology, then the specialization order of $\tau(\leq)$ is the order $\leq$ itself, i.e.; $\leq_{\tau(\leq)} = \leq$.

On the other hand, if (X, τ) is a T_o-Alexandroff space with the specialization order $\leq_\tau$, then the induced topology by the specialization order is the original topology,i.e.; $\tau(\leq_\tau) = \tau$ [2]. Therefore, T_o-Alexandroff spaces are completely determined by their specialization orders.

Note 2.1.1. See for example: Example 2.2.5 and Example 2.2.6.

Here is a general definition for the specialization order of a topological space:

Definition 2.1.2. [10] For the topological space (X, τ) we define the *specialization order* $\leq_\tau$ on X, for any $x, y \in X$, by

$$x \leq_\tau y \text{ if and only if } \forall\ O \in \tau,\ x \in O \text{ implies } y \in O.$$

Here, we are interested in Alexandroff spaces that satisfy the separation axiom T_o. We use their specialization orders in proofs to illustrate the results and the concepts. The importance of this study comes from the fact that we can characterize topological properties just by looking at its specializing order (poset).

For example, if we define a topological space X to be submaximal if each dense subset is open, then for the T_o-Alexandroff space X, X is submaximal if each element in the corresponding poset -the space X after being ordered by the specialization order $\leq_\tau$- is either maximal or minimal, i.e., the graph of its corresponding poset contains two rows; the row of the maximal elements and the row of the minimal elements([5]).

2.2 A glimpse of Alexandroff space

Now, let us go forward to study this class of topological spaces. First of all, let's recall the definitions of the neighborhood and nhood base of a point x in a space X, namely Definition 1.3.17 and Definition 1.3.18.

Remark 2.2.1. If the intersection of all the nhoods of $x \in X$ exists, then the set $V(x) = \bigcap\limits_{U\in\tau, x\in U} U$ is the smallest basic nhood of x (since, if W is any open set such that $x \in W$, then $V(x) = \bigcap\limits_{U\in\tau, x\in U} U \subseteq W$). So, the collection $\mathcal{B}_x$ with only one element, $\mathcal{B}_x = \{V(x)\}$, will be called the *minimal nhood base* of x.

Definition 2.2.2. [5] Let X be a T_0-Alexandroff space. Then, for each $x \in X, \uparrow x$ or $V(x)$ will denote the *minimal basic nhood* of x.

Definition 2.2.3. [14] An *Alexandroff space* X is a space in which any arbitrary intersection of open sets is open.

Remark 2.2.4. An equivalent statement of that in the above definition is that: an *Alexandroff space* X is a space in which each singleton has a minimal nhood base.

Proof. ($\Rightarrow$)Suppose that arbitrary intersection of open sets is open. That is; $\bigcap U \in \tau$ for all $U \in \tau$. Let $\mathcal{B}_x = \{U \in \tau : x \in U\}$ be a nhood base of $x \in X$. Then, $V(x) = \bigcap\limits_{U\in\tau, x\in U} U$ is open. So, $V(x) \subseteq U$ for all $U \in \mathcal{B}_x$. Thus, $\acute{\mathcal{B}}_x = \{V(x)\}$ is the minimal nhood base of x.

($\Leftarrow$) Suppose that for each $x \in X$, x has a minimal nhood base. That is; $\forall\ x \in X$, $\exists\ \mathcal{B}_x = \{V(x)\}$, a minimal nhood base of x. Let $\{U_\alpha : \alpha \in \Delta\}$ be the collection of all open sets in X. Let $y \in \bigcap_{\alpha\in\Delta} U_\alpha$. Then, $y \in U_\alpha$ for all $\alpha \in \Delta$. So, $y \in V(y) \subseteq U_\alpha$ for all $\alpha \in \Delta$. Therefore, $V(y) \subseteq \bigcap_{\alpha\in\Delta} U_\alpha$. Thus, $y \in V(y) \subseteq \bigcap_{\alpha\in\Delta} U_\alpha$. So, this intersection is open. Hence, arbitrary intersection of open sets is open. $\square$

Example 2.2.1. *It is easy to check that the discrete topology on any non-empty set is Alexandroff.*

Lemma 2.2.5. *[5] Any finite space is Alexandroff.*

Proof. It is obvious that any finite space is Alexandroff since any finite space has finite number of subsets and consequentially finite number of open sets. So, arbitrary intersections of these finite number of open sets is open. $\square$

Definition 2.2.6. [5] A topological space (X, τ) is called *locally finite* if each element x of X is contained in a finite open set and a finite closed set.

Proposition 2.2.7. *[5] Any finite space is locally finite.*

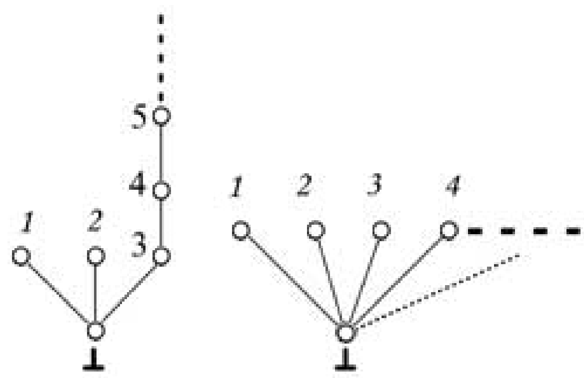
Proof. Let (X, τ) be a finite space. Since X is both open and closed set, then each element in X is contained in a finite open set and a finite closed set. Hence, X is a locally finite space. $\square$

Corollary 2.2.8. *[5] Each T_0-locally finite space is a T_0-Alexandroff space.*

Proof. It is enough to show that every locally-finite space is Alexandroff space. So, let X be a locally finite space. Then, each $x \in X$ is contained in an open set and hence has a basic nhood. That is; x has a minimal nhood base. $\square$

The converse of Corollary 2.2.8 need not be true. Here is a counterexample:

Example 2.2.2. *Let $P = \mathbb{N} \cup \{\perp\}$ and let P be ordered as indicated in any of the two diagrams. Clearly, P is a poset. Then, the T_0-topology defined on P with* *minimal base $\mathcal{B} = \{\uparrow x: \ x \in P\}$ is a T_0-Alexandroff (see Note 2.1.1) which is not a T_0-locally finite, since (see diagram (a)) for each $a \in P$ such that $3 \leq a$, $\uparrow a$ is not finite. Also (as in diagram (b)), $\uparrow \perp = P$ which is not finite.*



(a) (b)

It is worth noting that, there is a definition of similar name - locally finite- on a poset, but before introducing it, we have the following definition:

Definition 2.2.9. [22] Let P be a poset. A subset I of P is called a *poset interval*, or simply an *interval* if there exist elements $a, b \in P$ such that

$$I = \{t \in P: \ a \leq t \leq b\} = [a, b].$$

The elements a, b are called the endpoints of I. Clearly $a, b \in I$. Also, the endpoints of a poset interval are unique. That is; if $[a, b] = [c, d]$, then $a = c$ and $b = d$ (To see this: Let $I = \{t \in P: \ a \leq t \leq b\} = [a, b] = [c, d]$.) Since $c, d \in I$, then $a \leq c$ and $d \leq b$. Similarly, we have $c \leq a$ and $b \leq d$. Thus, $a = c$ and $b = d$).

Remark 2.2.10. [22] It is easy to see that the name is derived from that of an interval on a number line. From this analogy, one can easily define poset intervals without one or both endpoints. Whereas an interval on a number line is linearly ordered, a poset interval in general is not.

Definition 2.2.11. [22] A poset P is called *locally finite* if every interval $[x, y]$ in P is finite.

Corollary 2.2.12. *Every finite poset is locally finite.*

The converse of this corollary need not be true. The following example is a counterexample:

Example 2.2.3. *The set $\mathbb{Z}$ of integers with the usual order is a locally finite poset but not finite, while $\mathbb{Q}$ is neither.*

Example 2.2.4. *Recall Example 2.2.2. It is clear that the poset P is locally finite as a poset but not locally finite as a space.*

Definition 2.2.13. [5] A T_o-Alexandroff space whose corresponding poset satisfy the ACC is called *Artinian T_o-Alexandroff spaces.*

Throughout this section, the symbol $(X, \tau(\leq))$ denotes the T_o-Alexandroff space where $\leq$ is its (Alexandroff) specialization order.

Now, it is time to be more closer to the T_o-Alexandroff spaces.

Theorem 2.2.14. *[5] If $(X, \tau(\leq))$ is a T_o-Alexandroff space then a subset A of X is open if and only if it is an up set with respect to the specialization order; that is, A is open if and only if $A = \uparrow A$. And A is closed if and only if it is a down set; that is, A is closed if and only if $A = \downarrow A$.*

Proof. (1) For the first statement:
($\Rightarrow$) Suppose that $A \in \tau(\leq)$ and let $x \in A$. Let $y \in X$ such that $x \leq y$. Since $\mathcal{B} = \{\uparrow x : x \in X\}$ is the minimal nhood base of x, then $x \in \uparrow x \subseteq A$. But $y \in \uparrow x$, so $y \in A$. Thus, A is an up set.
($\Leftarrow$) Suppose that A is an up set. Let $x \in A$. Since A is up set, then we have $x \in \uparrow x \subseteq \uparrow A = A$. Since $\uparrow x \in \mathcal{B} = \{\uparrow x : x \in X\}$, then $\uparrow x$ is open and so do A.
(2) For the second statement:
Claim: For an open set B, $\uparrow B = (\downarrow B^c)^c$,

Proof of the claim: Let $x \in B^c$ and let $y \in X$ such that $y \leq x$. By the specialization order on X, $y \in \overline{\{x\}} \subseteq B^c$ (since B^c is closed). Thus, $y \in B^c$. Therefore, B^c is a down set and hence $B^c = \downarrow B^c$. Thus, $(\downarrow B^c)^c = (B^c)^c = B = \uparrow B$ and hence the claim.

Now, A is closed if and only if A^c is open if and only if $A^c = \uparrow A^c = (\downarrow A)^c$ (by the claim) if and only if $A = \downarrow A$. $\square$

Proposition 2.2.15. *[14] An Alexandroff topology $(X, \tau(\leq))$ is a T_0-space if and only if $x \neq y$ in X implies $V(x) \neq V(y)$.*

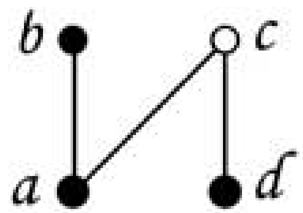
Proof. ($\Rightarrow$) If $x \neq y$, there exists an open set U that contains one and not the other. If $x \in U$, then $x \in V(x) \subseteq U$ and $y \notin V(x)$. Thus, $V(x) \neq V(y)$. Similarly if $y \in U$. ($\Leftarrow$) Let $x \neq y$. So, $V(x) \neq V(y)$. Suppose that $x, y \in V(x)$, then $x \notin V(y)$. To see this, suppose to contrary that $x \in V(y)$. Then, $x, y \in V(x) \cap V(y)$. Hence, $V(x) \cap V(y)$ is a nhood of y which is a proper subset of $V(y)$ which contradicts the fact that $V(y)$ is the minimal basic nhood containing y. $\square$

Suppose that A is a subset of a T_o-Alexandroff space $(X, \tau(\leq))$. Then two types of topologies are induced on A. One is the T_o-Alexandroff space on A with respect to the induced order $\leq$, and the other one is the induced topology $\tau(\leq)\mid_A$ which makes A as a subspace. It is not difficult to see that the two types coincide (see Theorem 2.2 of [2]).

Example 2.2.5. *Let $X = \{a, b, c, d\}$ with the partial order $a \leq b$, $a \leq c$ and $d \leq c$ as shown in the figure. Then, the T_o-Alexandroff topology is*

$$\tau = \{\phi, X, \{a, b, c\}, \{b\}, \{c\}, \{d, c\}, \{b, c, d\}, \{b, c\}\}$$

with minimal base $\mathcal{B} = \{\{a, b, c\}, \{b\}, \{c\}, \{d, c\}\}$. The set $A = \{a, b, d\}$ is a down *set which is not up set, so it is closed and not open. (Note that $A^c = \{c\} \in \tau$).*

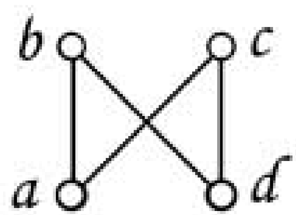


In the above example, from the given partial order we could graph the diagram of the poset and hence we could determine the base of the required topology and therefore we can write down all the members of the topology.

Example 2.2.6. *Let $X = \{a, b, c, d\}$, with the T_o-Alexandroff topology*

$$\tau = \{\phi, X, \{a, b, c\}, \{b\}, \{c\}, \{b, d, c\}, \{b, c\}\}$$

We can find the specialization order as follows: the closed sets are $X, \phi, \{d\}, \{a, c, d\}, \{a, b, d\}, \{a\}, \{a, d\}$. Now, $\overline{\{a\}} = \{a\}$, $\overline{\{d\}} = \{d\}$, $\overline{\{b\}} = \{b, a, d\}$ and $\overline{\{c\}} = \{c, a, d\}$, so (see the relation 2.1.1), $a \leq b$, $d \leq b$, $a \leq c$ and $d \leq c$ and hence the figure of the poset X is as shown in the following diagram:



Definition 2.2.16. [5] If $(X, \tau(\leq))$ is an Artinian T_o-Alexandroff space, we define M to be *the set of all maximal elements* of X. For a point $x \in X$, we define $\widehat{x} = \uparrow x \cap M$. The point x is *isolated* in X if $\{x\}$ is an open set, and hence maximal in X. So, M is the set of all isolated points in X. If A is a subset of an Artinian T_o-Alexandroff space, then we define $M(A)$ to be the set of all maximal elements of A under the induced order.

Example 2.2.7. *In the last two examples, Example 2.2.5 and 2.2.6,* $M = \{b, c\}$*. In Example 2.2.5,* $M(A) = \{b, d\}$*.*

Lemma 2.2.17. *[5] If* A *is open then* $\widehat{x} \subseteq A \ \forall\ x \in A$*, and if* $M(A) \nsubseteq M$ *then* A *is not open.*

Proof. Suppose that A is open and let $x \in A$. Since A is an up set and $x \in A$, then $\uparrow x \subseteq A$. Therefor, $\widehat{x} = \uparrow x \cap M \subseteq \uparrow x \subseteq A$.
For the second statement, suppose that $M(A) \nsubseteq M$ and A is open. Then, $\exists\ x \in M(A)$ such that $x \notin M$. Since $x \in M(A)$, and A is open then there isn't any $y \in X$ such that $x \leq y$. That is, $x \in M$ which is a contradiction. □

2.3 Identification of Basic Topological Concepts

To be more closer to the Alexandroff topology we must have a close look at the topological concepts applied on posets that occupied by Alexandroff topology.

Definition 2.3.1. [5] Let A be a subset of a T_o-Alexandroff space X. A point x is a *cluster point* of A if any $V(x)$ intersects $A - \{x\}$. The point $x \in A$ is *isolated* if $\{x\}$ is open in the subspace A; i.e., if $x \notin A'$, where A' is the set of all limit (or cluster) points of A.

Proposition 2.3.2. *[5] Let* $(X, \tau(\leq))$ *be a* T_o*-Alexandroff space and let* $A \subseteq X$*.*
(1) For $x \in X$, $\overline{\{x\}} = \downarrow x$*.*
(2) $A^o = \{x \in A : \uparrow x \subseteq A\}$*.*
(3) $\overline{A} = \bigcup_{x \in A} \downarrow x$*.*
(4) $A' = \overline{A} \backslash \{x : x \text{ is maximal in } A\}$*.*

Proof. (1) $\downarrow x$ is a down set and hence a closed set containing x. So, $\overline{\{x\}} \subseteq \downarrow x$. Conversely, let $y \in \downarrow x$, so $y \leq x$. If $y \in \overline{\{x\}}^c$, which is open set then for any $w \in X$ such that $y \leq w$, $w \in \overline{\{x\}}^c$. Therefore, $\uparrow y \cap \overline{\{x\}} = \phi$. Since $x \in \uparrow y$, we get that $x \notin \overline{\{x\}}$, which is a contradiction.

(2) Let $x \in \{x \in A : \uparrow x \subseteq A\}$. So, $\uparrow x \subseteq A$. But $\uparrow x$ is open, therefore $\uparrow x \subseteq A^o$. Thus, $x \in A^o$. Conversely, let $y \in A^o \subseteq A$. Since $A^o = \uparrow A^o$, then, $A^o \subseteq \uparrow A$. So, $y \in \{x \in A : \uparrow x \subseteq A\}$.

(3) If $x \in A$, then $\overline{\{x\}} = \downarrow x \subseteq \overline{A}$. So $\bigcup_{x \in A} \downarrow x \subseteq \overline{A}$. On the other hand, if $x \in A$ then $x \in \downarrow x \subseteq \bigcup_{x \in A} \downarrow x$. So $A \subseteq \bigcup_{x \in A} \downarrow x$, which is a closed set. Therefor $\overline{A} \subseteq \bigcup_{x \in A} \downarrow x$.

(4) If $x \in A'$ then $x \in \overline{A}$ (since $\overline{A} = A \cup A'$) and $\uparrow x \cap A \setminus \{x\} \neq \phi$ (since $\uparrow x$ is an open set containing x and x is a cluster point of A), so x is not maximal in A. Thus, $x \in \overline{A} \backslash \{x : x \text{ is maximal in A}\}$ and hence $A' \subseteq \overline{A} \backslash \{x : x \text{ is maximal in A}\}$. For the other inclusion, suppose that $y \in \overline{A}$, and y is not maximal in A. Then, we have that $\uparrow y \cap A \neq \phi$. If $\uparrow y \cap A = \{y\}$, then y is an isolated point and hence y is maximal in A, and this is not true (contradicts the assumption). So, we must have $y \in A'$. □

Theorem 2.3.3. *[5] Let $(X, \tau(\leq))$ be an Artinian T_o-Alexandroff space and let $A \subseteq X$. Then*

(1) $A^o = \phi$ if and only if $A \cap M = \phi$.

(2) $\overline{A} = \bigcup_{x \in M(A)} \downarrow x = \downarrow M(A)$.

(3) $A' = \bigcup_{x \in M(A)} \downarrow x \ \backslash \{x\} = \downarrow M(A) \backslash M(A)$.

(4) The subset A is dense if and only if $M \subseteq A$.

(5) The subset A is nowhere dense if and only if $M \cap A = \phi$.

(6) If $\mid M \mid \ = 1$, then any subset is either dense or nowhere dense.

Proof. (1) ($\Rightarrow$) Suppose that $A^o = \phi$ and let $x \in A \cap M$. So, x is maximal of X in A and hence $\uparrow x = \{x\} \subseteq A$. Thus, the open set $\uparrow x = \{x\} \subseteq A^o$, which contradicting the assumption that $A^o = \phi$.

($\Leftarrow$) Suppose that $A \cap M = \phi$, and $y \in A^o$. So $\uparrow y \subseteq A$. Since X satisfies the ACC, we get a maximal element z in X such that $y \leq z$ and so $z \in \uparrow y \subseteq A$. Therefor, $z \in A \cap M$ and hence, $A \cap M \neq \phi$ which is a contradiction.

(2) If $x \in A$, then there exists a maximal element y in A such that $x \leq y$, so $\downarrow x \subseteq \downarrow y$, and this implies that $\overline{A} = \bigcup_{x \in A} \downarrow x \subseteq \bigcup_{x \in M(A)} \downarrow x = \downarrow M(A)$.

The other inclusion is obvious, since $\{\downarrow x : x \text{ is maximal in } A\} \subseteq \{\downarrow x : x \in A\}$.

(3) Since X satisfies the ACC, it follows that

$$\begin{aligned} A &= \overline{A} \backslash \{x : x \text{ is maximl in } A\} \\ &= \bigcup_{x \in M(A)} (\downarrow x) \ \backslash M(A) \\ &= \bigcup_{x \in M(A)} \downarrow x \ \backslash \{x\}. \end{aligned}$$

.

(4) ($\Rightarrow$) Suppose that A is dense, and let $x \in M$. Then $\uparrow x \cap A \neq \phi$ (by Theorem 1.3.19). But $\uparrow x = \{x\}$, so $x \in A$.

($\Leftarrow$) Suppose that $M \subseteq A$, so $M(A) = M$. By part (2),

$$\overline{A} = \bigcup_{x \in M(A)} \downarrow x = \bigcup_{x \in M} \downarrow x = X$$

(5)($\Rightarrow$) Suppose that A is nowhere dense, i.e., $\overline{A}^o = \phi$, so by part(1) of this Theorem, $M \cap \overline{A} = \phi$, and hence $M \cap A = \phi$.

($\Leftarrow$) Suppose that $M \cap A = \phi$, so no maximal element of X is in A. By Proposition 2.3.2, no maximal elements of X is in $\overline{A}$, and hence $\overline{A}^o = \phi$.

(6) Let $M = \{\top\}$, and let A be a subset of X. Then either $\top \in A$ or $\top \notin A$, and by parts (4) and (5) above, either A is dense or nowhere dense. $\square$

Chapter 3

Scott Topology

In this chapter, we introduce first the definition of the Scott topology followed by some illustrative examples of Scott open sets and Scott topology. Furthermore, we shed some light on some properties of the Scott topology such as being a T_0-space or a sober space. Also, we seek in the relation between the Scott topology and the Alexandroff topology. The base of this topology is given later.

3.1 Scott open sets

We begin this section by giving the definition of the Scott-open set.

Definition 3.1.1. [8] A subset U of a poset P is *Scott open* if:
(i) U is an up set (some authors use the term "upper set" instead of up set with the same definition), and
(ii) U is inaccessible by directed suprema. That is; for any directed subset $S \subseteq P$ with a supremum $\bigvee S$, if $\bigvee S \in U$, there exists $s_0 \in S$ such that $s_0 \in U$ (i.e., $S \cap U \neq \phi$).

In this case, $s \in U$ for each $s \in P$ with $s_0 \leq s$.

Remarks 3.1.2. (1) From (i) in the above definition, we observe that every Scott open is Alexandroff open.

(2) Some authors define the Scott topology over a dcpo and then S in (ii) of the above definition is supposed to be just directed (see [4]).

(3) A subset $F \subseteq P$ is Scott closed if its complement is Scott open. That is; if F^C is an up set and for any directed subset D of P that has a supremum $\bigvee D$ with $\bigvee D \in F^C$, we have $D \cap F^C \neq \phi$ and so $D \nsubseteq F$.

This leads us to the following lemma:

Lemma 3.1.3. *A subset F of a poset P is Scoot closed if it satisfies:*
(i) $F = \downarrow F$ (F is a down set) and
(ii) if D is a directed set contained in F and $\sup D$ exists, then $\sup D \in F$.

Proof. We get this result by the contrapositive of part (3) of Remarks 3.1.2, together with the fact that the complement of an up set is a down set. $\square$

Example 3.1.1. *Let $P = [0, 10]$, and define $\leq$ on P to be the usual order.*
It is clear that P is a dcpo. Let $U = (1, 10]$ be a subset of P (clearly U is an up set) and let S be any directed subset of P such that $\bigvee S \in U$. Therefore, $1 < \bigvee S \leq 10$. Clearly, $S \nsubseteq [0, 1)$(otherwise $\bigvee S \notin U$). Thus $S \cap U \neq \phi$. Hence, the set U is Scott open.

Lemma 3.1.4. *[17] If D is a dcpo, then the set $U_x = \{z \in D : \ z \nleq x\}$ is a Scott open set.*

We can generalis the above lemma as follows:

Lemma 3.1.5. *Let D be a poset. Then, the set $U_x = \{z \in D : \ z \nleq x\} = D - \downarrow x$ is a Scott open set.*

Proof. Let $z \in U_x$ and let $y \in D$ such that $z \leq y$. Suppose that $y \notin U_x$. Then, $y \leq x$ and hence, $z \leq x$. So, $z \notin U_x$ which is a contradiction. Hence, $y \in U_x$ and hence, U_x is an up set.

Now, let S be any directed subset of D such that $\bigvee S$ exists and $\bigvee S \in U_x$. Then, $\bigvee S \nleq x$. Assume that for any $s \in S, s \notin U_x$. So, $s \leq x$ for all $s \in S$. Thus, x is an upper bound of S and hence, $\bigvee S \leq x$ which is a contradiction. Thus, $S \cap U_x \neq \phi$. Hence, U_x is a Scott open. $\square$

Proposition 3.1.6. *[4] Let P be a dcpo and let K_P denote the set of all compact elements in P. Then, for any $a \in K_P, \uparrow a = \{x : a \leq x\}$ is Scott open.*

Proof. It is clear that $\uparrow a$ is an up set. Let S be any directed subset of P such that $\bigvee S \in \ \uparrow a$. So, $a \leq \bigvee S$. Since a is compact, then there is $s \in S$ such that $a \leq s$ and hence $\uparrow a \cap S \neq \phi$. Therefore, $\uparrow a$ is a Scott open. $\square$

Example 3.1.2. *The set of all real numbers $\mathbb{R}$, is Scott closed.*

Proof. Clearly $\mathbb{R}$ is a down set. If $D \subseteq \mathbb{R}$ is a directed subset and $\bigvee D$ exists, then $\bigvee D \in \mathbb{R}$. Therefore, $\mathbb{R}$ is closed. $\square$

Recalling that an ideal I is a directed-lower set (see Definition 1.2.20), we have:

Proposition 3.1.7. *In finite posets, every ideal is Scott closed.*

Proof. Let P be a finite poset and let I be an ideal subset of P. Since P is finite, then it is a dcpo. Since I is an ideal, then I is a directed-lower set. So, $I = \downarrow I$. Now, let D be any directed subset of I such that $\sup D$ exists (It does, since $D \subseteq P$ and P is a dcpo). Again by the finiteness of P and by Proposition 1.2.4, $\sup D \in D$ and hence $\sup D \in I$. This proves the desired result. $\square$

Proposition 3.1.8. *[4] Let P be a poset. Then, the collection of all Scott open sets forms a topology on P. This topology is called the Scott topology and is denoted by $\sigma(P)$.*

Proof. Let $\sigma(P)$ be the collection of all Scott open subsets of P. It is clear that both ϕ and P are Scott open.

Next, let $A, B \in \sigma(P)$ and let $x \in A \cap B$ and $x \leq y$. Therefore, $x \in A$, $x \in B$ and $x \leq y$. Since A, B are up sets, $y \in A$ and $y \in B$. That is; $y \in A \cap B$.

Now, let S be any directed subset of P with supremum such that $\bigvee S \in A \cap B$. Therefore, $\bigvee S \in A$ and $\bigvee S \in B$. Since both A and B are Scott open, then $\exists s_1, s_2 \in S$ such that $s_1 \in A$ and $s_2 \in B$. Since S is directed, then $\exists s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$ and hence, $s \in A$ and $s \in B$ (by hereditability of A and B). Hence, $s \in A \cap B$ and hence $A \cap B \in \sigma(P)$.

Finally, let $\{A_\alpha : \alpha \in \Delta\}$ be a family of Scott open subsets of P. Let $x \in \bigcup_{\alpha \in \Delta} A_\alpha$ with $x \leq y$. Therefore $x \in A_\alpha$, for some $\alpha \in \Delta$. Since A_α is Scott open and $x \leq y$, then $y \in A_\alpha \subseteq \bigcup_{\alpha \in \Delta} A_\alpha$.

Now, let S be a directed subset of P with supremum $\bigvee S \in \bigcup_{\alpha \in \Delta} A_\alpha$. Then, $\bigvee S \in A_\beta$ for some $\beta \in \Delta$. But A_β is Scott open, so $\exists s \in S$ such that $s \in A_\beta \subseteq \bigcup_{\alpha \in \Delta} A_\alpha$. This completes the proof. $\square$

Example 3.1.3. *The right ray topology on $\mathbb{R}$ is the Scott topology of usual order.*

Proof. Consider $\mathbb{R}$ with its usual order and let τ_r denote the right ray topology on $\mathbb{R}$, that is; $\tau_r = \{\phi, \mathbb{R}\} \cup \{(x, \infty) : x \in \mathbb{R}\}$. Clearly, for any $x \in \mathbb{R}$, $U = (x, \infty) \in \tau_r$ is an up set. Now, let S be any directed subset of $\mathbb{R}$ with supremum $\bigvee S$ exists such that $\bigvee S \in U$. Thus, $x < \bigvee S < \infty$ and hence, x is not an upper bound of S. So, there is $s \in S$ such that $x < s \leq \bigvee S$. So, $s \in U$ and so, $U \cap S \neq \phi$. Hence, U is Scott open. $\square$

3.2 Scott Topology and Alexandroff Topology

One consequence of the definition of the Scott topology is that, any algebraic dcpo D (see Definition 1.2.11) is completely determined by its Scott topology.

Lemma 3.2.1. *[4] The order relation on an algebraic dcpo D can be completely recovered by setting $x \leq y$ if and only if $\forall\ O \in \sigma(D)$, $x \in O$ we have that $y \in O$.*

Proof. ($\Rightarrow$) It is obvious since O is an up set.

($\Leftarrow$) To prove the other implication, one should note that for any compact element $a \in K_D$, $\uparrow a = \{x \in D : a \leq x\}$ is a Scott open (see Proposition 3.1.6). So, by the hypothesis that $\forall\ O \in \sigma(D)$, $x \in O \Rightarrow y \in O$, choosing $a \in K_D$ such that $x \in \uparrow a$ implies that $y \in \uparrow a$. Hence $a \leq x \Rightarrow a \leq y$ and so, by Proposition 1.2.15, this is equivalent to $x \leq y$. □

Corollary 3.2.2. *[4] On any algebraic dcpo D, the Scott topology $\sigma(D)$ is T_o.*

Proof. It is enough to show that, if $\forall\ O \in \sigma(D)$, the condition that " $x \in\ O$ if and only if $y \in O$" implies that $x = y$. Now, by the above lemma, the condition " $x \in\ O$ if and only if $y \in O$" implies that $x \leq y$ and $y \leq x$ and hence, $x = y$. □

Lemma 3.2.3. *Every Scott-open set is a Alexandroff open. That is; on any poset P, the Scott topology is coarser than the Alexandroff topology.*

Proof. The proof follows immediately from Definition 3.1.1(i) and Theorem 2.2.14. □

The following examples are an illustrative examples. We show that the Scott topology on a poset may be proper subcollection of the T_0-Alexandroff topology.

Example 3.2.1. *Consider the set $\mathbb{R}$ of real numbers with the usual order. Let $\mathcal{B}_a = \{[x, \infty) : x \in \mathbb{R}\}$. It is clear that $\mathcal{B}_a$ is a base for the T_0-Alexandroff topology on $\mathbb{R}$. Now, let $\sigma(\mathbb{R})$ denote the right ray topology on $\mathbb{R}$. Then, $\sigma(\mathbb{R})$ is the Scott topology on $\mathbb{R}$ (see Example 3.1.3). That is; $\sigma(\mathbb{R}) = \{\phi, \mathbb{R}\} \cup \{(y, \infty) : y \in \mathbb{R}\}$. Since $(y, \infty) = \bigcup_{n \in \mathbb{N}} [y + \frac{1}{n}, \infty) \in \tau_a$, where τ_a is the T_0-Alexandroff topology, then $\sigma(\mathbb{R}) \subseteq \tau_a$.*

Claim: $U = [x, \infty)$ is not Scott open. To see this, let $S = [b, x) \subseteq \mathbb{R}$, where $b < x$. Then S is a directed set and $\bigvee S = x\ \in U$. Moreover, $S \cap U = \phi$. Thus, U is not Scott open.

Example 3.2.2. *Recall Example 1.2.9. Let $U \subseteq D$ be such that $U = \{1, 3\}$. Clearly U is an up set and hence an Alexandroff-open.*

Claim: U is not Scott open. For, if $S = [2, 3)$, then S is a directed subset of D with $\bigvee S = 3 \in U$ and $S \cap U = \phi$. Hence, U is not Scott open.

Proposition 3.2.4. *[21] On a finite poset D, the T_0-Alexandroff topology agrees with the Scott topology. That is; $\tau(\leq) = \sigma(D)$.*

Proof. Let D be a finite poset. Therefore, D is a dcpo (see Example 1.2.3). Let U be any Alexandroff-open set and let S be any directed subset of D such that $\bigvee S \in U$. Since S is directed and D is finite, by Proposition 1.2.4, S has a top element $\top = \bigvee S \in S$. So, $U \cap S \neq \phi$ and hence, U is Scott open. □

Proposition 3.2.5. *[4] Let D be an algebraic dcpo. Then, the family*

$$\uparrow K_D = \{\uparrow a : a \in K_D\}$$

is a base for the Scott topology $\sigma(D)$ on D.

Proof. For any $x \in D$, there exists a compact element $a \in K_D$ such that $a \leq x$, that is; $x \in \uparrow a$ (see Lemma 1.2.12). So, $D \subseteq \bigcup_{a \in K_D} \uparrow a$ and hence, $D = \bigcup_{a \in K_D} \uparrow a$.

Next, let $x \in \uparrow a \cap \uparrow b$ for $a, b \in K_D$. So, $a, b \in \downarrow_K x$. Since $\downarrow_K x$ is directed, then there exists $c \in \downarrow_K x$ such that $a \leq c$ and $b \leq c$. This implies that $x \in \uparrow c \subseteq \uparrow a \cap \uparrow b$. Hence, by Theorem 1.3.16, $\uparrow K_D$ is a base for $\sigma(D)$. □

Example 3.2.3. *Let P be as in Example 1.2.5. Then $P = K_P$ and P is algebraic dcpo. So, $\mathcal{B} = \{\uparrow a : a \in K_P\} = \{\uparrow x : x \in P\}$ is a base for the Scott topology on P. It is clear that the Scott topology on P is the same as the T_0-Alexandroff topology.*

The following property is an interesting and a strong compactness property.

Lemma 3.2.6. *[4] Let D be a poset. Then, for any $U \subseteq D$ and for any $a \in D$,*

$$\uparrow a \subseteq \bigcup_{b \in U} \uparrow b \text{ if and only if } \exists\, b \in U \text{ such that } \uparrow a \subseteq \uparrow b.$$

Proof. (⇒) Suppose that $\uparrow a \subseteq \bigcup_{b\in U} \uparrow b$. Then $a \in \bigcup_{b\in U} \uparrow b$ (since $a \in \uparrow a$). Therefore, $\exists\, b \in U$ such that $a \in \uparrow b$ (and so, $b \leq a$) which gives that $\uparrow a \subseteq \uparrow b$.

(⇐) Obvious. □

Lemma 3.2.7. *[4] Let D be an algebraic dcpo.Then, any Scott open $O \subseteq D$ is the union of the basic open sets such that*

$$O = \bigcup_{a\in O\cap K_D} \uparrow a.$$

Proof. Let O be any Scott open.

($\subseteq$) Let $x \in O$. Then (by Lemma 1.2.12) $\exists a \in K_D$ such that $a \leq x$. That is; $x \in \uparrow a$. Thus, $x \in \uparrow a \cap O \subseteq \uparrow a$ and hence $x \in \bigcup_{a\in O\cap K_D} \uparrow a$.

($\supseteq$) Let $x \in \bigcup_{a\in O\cap K_D} \uparrow a$. Therefore, $x \in \uparrow a$ for some $a \in O \cap K_D$. That is; $a \leq x$ for some $a \in O \cap K_D \subseteq O$. Since O is a Scott open, then $x \in O$. Thus, $\bigcup_{a\in O\cap K_D} \uparrow a \subseteq O$. □

What about the subspace of a Scott space?. Here is the answer:

Proposition 3.2.8. *A subspace of a Scott topology is a Scott subspace.*

Proof. Let $(X, \sigma(X))$ be a Scott topological space and let A be any subset of X. Then, A has the relative topology $T_A = \{A \cap U : U \in \sigma(X)\}$. Let $B \in T_A$. So, there exists $U \in \sigma(X)$ such that $B = A \cap U$. Let $x \in B$ and $y \in A$ such that $x \leq y$. Since $x \in U \in \sigma(X)$, then $y \in U$ and hence $y \in A \cap U = B$. Thus, B is an up set with respect to A. Now, let S be any directed subset of A such that $\sup S$ exists and $\sup S \in B$. Then, $S \cap A = S \neq \phi$. Since U is Scott open, then $S \cap U \neq \phi$. Therefore, $S \cap B = S \cap (A \cap U) = (S \cap A) \cap U = S \cap U \neq \phi$. Hence B is a Scott-open with respect to A. □

The following theorem is a generalization of Corollary 3.2.2:

Theorem 3.2.9. *On any poset, the Scott topology is T_0.*

Proof. Let X be a poset and let $x, y \in X$ such that $x \neq y$. Then either $x \notin\downarrow y$ or $y \notin\downarrow x$. Suppose that $x \notin\downarrow y$. Then, $x \in (\downarrow y)^c$ and $y \notin (\downarrow y)^c$. By Lemma 3.1.5, $(\downarrow y)^c$ is Scott open. Similarly if $y \notin\downarrow x$. So, we are done. $\square$

Note that the Scott topology is not T_1, since, WLOG, if $x < y$, then any up set contains x must contain y.

Now, we conclude the current chapter by giving the definition of the "sober topology", but before going forward we have some thing to introduce.

Definition 3.2.10. [4] Given a topological space (X, τ) and any base $\mathcal{B}_\tau$ for τ, we define the set $\mathrm{Pt}(\mathcal{B}_\tau)$ of *formal points* of the topology consisting of elements of non-empty subcollections α of $\mathcal{B}_\tau$ such that:

(1) $\phi \notin \alpha$. (That is; $U \neq \phi, \forall\, U \in \alpha$).

(2) For any $U,\ V \in \alpha$, there exists $W \in \mathcal{B}_\tau$ such that $W \in \alpha$ and $W \subseteq U \cap V$.

(3) For any $U \in \alpha$ such that $U \subseteq \bigcup\limits_{\{V_i \in \mathcal{B}_\tau:\ i \in \Delta\}} V_i$, $\exists\ i \in \Delta$ such that $V_i \in \alpha$.

Note that condition (2) above insures that for any $U, V \in \alpha$, $U \cap V \neq \phi$. Moreover, the V_i's in condition (3) are elements of $\mathcal{B}_\tau$.

Example 3.2.4. *Consider $\mathbb{R}$ with the standard topology τ. Then, $\mathcal{B}_\tau = \{(a, b) :\ a, b \in \mathbb{R},\ a < b\}$ forms a base for τ. For each $x \in \mathbb{R}$, let $\alpha_x = \{(x - \varepsilon, x + \delta) : \varepsilon, \delta > 0\}$. Then α_x is a formal point. To see this:*

(1) Clearly $\alpha_x \subseteq \mathcal{B}_\tau$ and $\phi \notin \alpha_x$.

(2) Let $U, V \in \alpha_x$ such that $U = (x - \varepsilon_1, x + \delta_1)$ and $V = (x - \varepsilon_2, x + \delta_2)$. Now, $U \cap V = (x - \varepsilon, x + \delta)$ where $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and $\delta = \min\{\delta_1, \delta_2\}$. Set $W = U \cap V = (x - \varepsilon, x + \delta)$. Then, $W \in \mathcal{B}_\tau, W \in \alpha_x$ and $W \subseteq U \cap V$.

(3) Let $U \in \alpha_x$ such that $U = (x - \varepsilon, x + \delta)$. Now, $U \subseteq \bigcup\limits_{\{V_i \in \mathcal{B}_\tau:\ i \in \Delta\}} V_i$ implies that

$x \in V_i = (a_i, b_i)$ *for some* $i \in \Delta$. *Set* $\varepsilon_i = x - a_i > 0$, $\delta_i = b_i - x > 0$. *So,* $V_i = (x - \varepsilon_i, x + \delta_i) \in \alpha_x$.

Now, we will define the canonical map by means of the formal points (since this map will be our tool to define the sober Scott topology), but after proving the following lemma:

Lemma 3.2.11. *[4] Let* (X, τ) *be a topological space and let* $\mathcal{B}_\tau$ *be a base for* τ. *Then, the collection* $\mathrm{B}_x = \{U \in \mathcal{B}_\tau : \ x \in U\}$ *of basic nhoods forms a formal point for each* $x \in X$.

Proof. Let us denote the set $\{U \in \mathcal{B}_\tau : x \in U\}$ by B_x. We will show that B_x is a formal point. Clearly $\mathrm{B}_x = \{U \in \mathcal{B}_\tau : x \in U\} \subseteq \mathcal{B}_\tau$. Since $x \in U \ \forall \ U \in \mathrm{B}_x$, then $U \neq \phi \ \forall \ U \in \mathrm{B}_x$. Now, Let $U, V \in \mathrm{B}_x$. Then, $x \in U \cap V$. Since $\mathcal{B}_\tau$ is a base, then there exists $W \in \mathcal{B}_\tau$ such that $x \in W \subseteq U \cap V$. So, $x \in W$. Thus, $W \in \mathrm{B}_x$. Finally, Let $U \in \mathrm{B}_x$ such that $U \subseteq \bigcup_{i \in \Delta} V_i$, where $V_i \in \mathcal{B}_\tau \ \forall \ i$. Since $U \in \mathrm{B}_x$, then $x \in U$. That is; $x \in U \subseteq \bigcup_{i \in \Delta} V_i$. So $x \in V_i$ for some $i \in \Delta$ which implies that $V_i \in \mathrm{B}_x$. $\square$

Definition 3.2.12. [4] For any topological space (X, τ), the *canonical map* $\phi : X \to \mathrm{Pt}(\mathcal{B}_\tau)$ is defined by putting:

$$\phi(x) = \mathrm{B}_x = \{U \in \mathcal{B}_\tau : x \in U\}.$$

The canonical map has the following property:

Lemma 3.2.13. *[4] Let* (X, τ) *be a topological space and let* $\phi : X \to Pt(\mathcal{B}_\tau)$ *be the canonical map. If* τ *is* T_0, *then the map* ϕ *is injective.*

Proof. $\forall \ x, y \in X$ such that $\phi(x) = \phi(y)$ we have, $\forall \ U \in \mathcal{B}_\tau, x \in U \leftrightarrow y \in U$. Since τ is T_0, then $x = y$. Hence, ϕ is 1-1. $\square$

Definition 3.2.14. [4] Let (X, τ) be a topological space and let $\phi : X \to \mathrm{Pt}(\mathcal{B}_\tau)$ be the canonical map defined above. If ϕ is bijective, then τ is called *sober*. In this case, no new formal point is added in $\mathrm{Pt}(\mathcal{B}_\tau)$ which is not already an image of a point in X.

Proposition 3.2.15. *[4] For any algebraic dcpo $(D, \leq)$, the Scott topology $\sigma(D)$ with base $\uparrow K_D$ is sober. In this case, the formal points are the non-empty subcollections α of $\uparrow K_D$ such that:*

(i) $\forall \uparrow a, \uparrow b \in \alpha, \exists\ c \in K_D$ such that $\uparrow c \in \alpha$ and $a \leq c$ and $b \leq c$.

(ii) For any $a, b \in D$ such that $\uparrow a \in \alpha,\ b \leq a$, we have $\uparrow b \in \alpha$.

Proof. [4] The first condition on formal points holds since all the elements of $\uparrow K_D$ are non-empty. (To see this, let $\uparrow a \in \uparrow K_D$, then $a \in K_D$. So for any directed subset U of D such that $a \leq \bigvee U$, $\exists\ u \in U$ such that $a \leq u$. So, $u \in \uparrow a$. That is; $\uparrow a \neq \phi$). Moreover, for any $a, b \in K_D, \uparrow a \subseteq \uparrow b$ if and only if $b \leq a$ (note that $\uparrow a, \uparrow b$ are Scott open and hence up sets) and hence the first condition in the proposition, (in (i)), is just re-writing of the second condition on the formal points' definition of a generic topological space. That is, we can restate the first condition here as follows: for any $\uparrow a, \uparrow b \in \alpha, \exists\ \ \uparrow c \in K_D$ such that $\uparrow c \in \alpha$ and $\uparrow c \subseteq (\uparrow a \cap \uparrow b)$, so $a \leq c$ and $b \leq c$. This shows that (i) of the proposition holds.
The third condition in the definition of the formal points is here substituted by a simpler one, because of the strong compactness property (Lemma 3.2.6) applied on the base $\uparrow K_D$. (To see this, let $a, b \in D$ such that $\uparrow a \in \alpha,\ b \leq a$. Then,$\uparrow a \subseteq \uparrow b$ and hence $\uparrow b$ is a cover for $\uparrow a$. So, by the compactness property of $K_D, \uparrow b \in \alpha$). This shows that (ii) of the proposition holds.
Now, observe that, for any formal point α, the subset $U_\alpha = \{a \in K_D |\ \uparrow a \in \alpha\}$ is directed, because: if $a, b \in U_\alpha$, then $\uparrow a, \uparrow b \in \alpha$. So, there exists $c \in K_D$ such that $\uparrow c \in \alpha$. Hence, $c \in U_\alpha$ such that $a \leq c$ and $b \leq c$ (from (i) here). This shows that U_α is directed. Since D is a dcpo and U_α is directed, $\bigvee U_\alpha$ exists.
Since the Scott topology is a T_o-space on D, then the canonical map ϕ is injective (by Lemma 3.2.13). So, to show that the Scott topology $\sigma(D)$ is sober we have only to show that ϕ is surjective.

Now, we can prove that the map $\phi : D \to \mathrm{Pt}(\uparrow K_D)$ is surjective by showing that: for any formal point α, $\alpha = \phi(\bigvee U_\alpha)$. To see this:

Let $a \in K_D$ such that $\uparrow a \in \phi(\bigvee U_\alpha)$. So, $a \leq \bigvee\limits_{\uparrow b \in \alpha} b$. Since a is a compact element and U_α is directed and

$$a \leq \bigvee_{\uparrow b \in \alpha} b = \bigvee U_\alpha$$

, we obtain that $\exists\, b \in K_D$ such that $\uparrow b \in \alpha (\equiv b \in U_\alpha)$ and $a \leq b$. That is, $\exists\, b \in K_D$ such that $\uparrow b \in \alpha$ and $\uparrow b \subseteq \uparrow a$, which shows that $\uparrow a \in \alpha$, since α is a formal point (by the second condition of its definition).

For the other inclusion, $\uparrow a \in \phi(\bigvee U_\alpha) = \mathrm{B}_{\bigvee U_\alpha}$ implies that $\bigvee U_\alpha \in \uparrow a$. This implies that $a \leq \bigvee U_\alpha$. Then,

$$\begin{aligned} \phi(\bigvee U_\alpha) &= \{U \in \uparrow K_D : \bigvee U_\alpha \in U\} \\ &= \{\uparrow b : \bigvee U_\alpha \in \uparrow b, b \in K_D\} \\ &= \{\uparrow b : b \leq \bigvee U_\alpha, b \in K_D\} \\ &= \alpha. \end{aligned}$$

This completes the proof of the proposition. Hence the Scott topology is sober. □

Chapter 4

Scott Topology and Approximation relation

In this chapter we introduce the definition of the approximation relation and some of its important properties. Then we will look to the Scott topology through this relation. The definition of the continuity of a poset will be introduced and through this we shall see some of the applications of the Scott topology.

4.1 The approximation relation

A significant contribution of the theory of continuous partially ordered sets has been the explicit definition and use of a new order relation, one that sharpens the traditional notion of order, namely the approximation relation. Here we introduce the definition of this relation and browse some of its properties.

Definition 4.1.1. [8] Let $(D, \leq)$ be a poset. Then, for any $x, y \in D$ we write $x \ll y$ if and only if for all directed sets $S \subseteq D$ with a supremum $\bigvee S$, $y \leq \bigvee S$ implies that there exists $s \in S$ such that $x \leq s$. In other words, $x \ll y$ if and only if every directed set with join above y has a member above x, [10].

For the symbol " $\ll$ ", read "approximates" (Some authors prefer the term "way-below"[3]). If $x \ll y$, we say x approximates y (or x is way-below y). When confusion may arise, the relation $\ll$ in a poset D will be specifically written $\ll_D$. We define $\Downarrow x = \{a \in D : a \ll x\}$ and $\Uparrow x = \{a \in D : x \ll a\}$.

Remark 4.1.2. The set $\Downarrow x$ (resp. $\Uparrow x$) is the set of all elements that approximate x (resp. that x approximates). In some texts (see [7]), $\Downarrow x$ (*resp.* $\Uparrow x$) is called the way- below (-above) set of x.

Example 4.1.1. *Let $P = [1, 3]$. Then, $(P, \leq)$- where $\leq$ is the usual order - is a dcpo. Clearly $1 \ll 2$ since for any directed subset U with $2 \leq \bigvee U$, there exists $u \in U$ such that $1 \leq u$. Actually, $1 \ll x$ for each $x \in P$.*

Some basic properties of the approximation relation are given in the following proposition:

Proposition 4.1.3. *[10] In any partially ordered set P,*

(1) $x \ll y$ implies $x \leq y$.

(2) $z \leq x \ll y \leq w$ implies $z \ll w$.

(3) If $\perp$ is the least element, then $\perp \ll x$, $\forall\, x \in P$.

Proof. (1) Suppose that $x \ll y$. Since $\{y\}$ is a directed subset with $\bigvee\{y\} = y$, $y \leq \bigvee\{y\} = y$ and $x \ll y$, then there exists $s \in \{y\}$ such that $x \leq s$. So, $x \leq s = y$.

(2) Suppose that $z \leq x \ll y \leq w$. Let S be any directed subset with supremum $\bigvee S$ such that $w \leq \bigvee S$. Since $y \leq w$, then $y \leq \bigvee S$. Now, $x \ll y$ and $y \leq \bigvee S$, then $\exists\, s \in S$ such that $x \leq s$. Finally, $z \leq x$ implies $z \leq s$. Therefore, $z \ll w$.

(3) Let $x \in P$ and let S be any directed subset with supremum $\bigvee S$ such that $x \leq \bigvee S$. Since S is directed, then $S \neq \phi$. So, there exists $s \in S$ such that $\perp \leq s$ (since $\perp \leq s$ for all $s \in P$) . Hence, $\perp \ll x$ for all $x \in P$. $\square$

The converse of part (1) of the above proposition need not be true as the following example shows.

Example 4.1.2. *Recall Example 1.2.9. One can easily check that D is a dcpo and $1 \leq 3$. Let $S = [2, 3)$, then S is a directed subset of D with $\bigvee S = 3$. Clearly $3 \leq 3 = \bigvee S$. But for any $x \in S, 1 \nleq x$. Hence, $1 \not\ll 3$.*

More properties for the approximation relation are given in the following proposition:

Proposition 4.1.4. *[3] Let $(D, \leq)$ be a poset. Then, the approximation relation on D has the following properties :*

(1) If $x \ll y$ and $y \ll x$, then $x = y$.

(2) If $x \ll y$ and $y \ll z$ then $x \ll z$.

(3) $x \in \Downarrow y$ if and only if $x \ll y$.

(4) $x \in \Uparrow y$ if and only if $y \ll x$.

(5) For any $x \in D$, $\Downarrow x \subseteq \downarrow x$ and $\Uparrow x \subseteq \uparrow x$.

(6) For any $x, y \in D$, if $x \leq y$ then $\Downarrow x \subseteq \Downarrow y$ and $\Uparrow y \subseteq \Uparrow x$.

Proof. (1) The proof follows immediately by applying (1) in Proposition 4.1.3 above together with the anti-symmetric property of $\leq$ on D.

(2) Let S be a directed subset of D with supremum $\bigvee S$ such that $z \leq \bigvee S$. Since $y \ll z$ then $\exists\ s \in S$ such that $y \leq s$. Since $x \ll y$ implies $x \leq y$, therefore $x \leq s$. That is; $\exists\ s \in S$ such that $x \leq s$. Hence, $x \ll z$.

(3),(4) Just re-stating the definitions.

(5) Let $z \in \Downarrow x$. Therefore, $z \ll x$ and hence $z \leq x$. That is, $z \in \downarrow x$. Hence, $\Downarrow x \subseteq \downarrow x$.

Now, let $w \in \Uparrow x$. Therefore, $x \ll w$ and hence $x \leq w$. That is, $w \in \uparrow x$. Hence, $\Uparrow x \subseteq \uparrow x$.

(6) Suppose that $x \leq y$ and let $z \in \Downarrow x$. So, $z \ll x$. Therefore, $z \leq z \ll x \leq y$.

Thus, by Proposition 4.1.3 part (2), $z \ll y$. Hence, $\Downarrow x \subseteq \Downarrow y$.

For the second statement, let $w \in \Uparrow y$. So, $y \ll w$. Now, $x \leq y \ll w \leq w$ implies that $x \ll w$. Hence, $\Uparrow y \subseteq \Uparrow x$. □

Lemma 4.1.5. *In a finite poset P, $x \leq y$ if and only if $x \ll y$. That is; $\downarrow y$ coincides with $\Downarrow y$.*

Proof. We will prove only the first direction. So, let P be a finite poset. Therefore, P is a dcpo. Now, Suppose that $x \leq y$ and let S be any directed subset of P such that $y \leq \bigvee S$. Therefore, $x \leq \bigvee S$. Since S has a top element $\top = \bigvee S \in S$, then, take $s = \top \in S$ so that $x \leq s$. Hence, $x \ll y$. □

The following example shows that the approximation relation need not be reflexive.

Example 4.1.3. *Recall Example 1.2.9 . Let $S = [2, 3)$. Clearly S is directed subset with $\bigvee S = 3$ and $1 \leq \bigvee S = 3$. Since $\forall x \in S$, $1 \| x$, then there is no $x \in S$ such that $1 \leq x$. Thus, $1 \not\ll 1$.*

Proposition 4.1.6. *[6] For any dcpo P, an element $k \in P$ is compact if and only if $k \ll k$.*

Proof. ($\Rightarrow$) Let P be any dcpo and let $k \in P$ be a compact element. Let U be any directed subset such that $k \leq \bigvee U$. Then, by the definition of the compact element, there is $u \in U$ such that $k \leq u$. That is, $k \ll k$.

($\Leftarrow$) Straightforward from the definitions of the approximation relation and the compact element. □

Due to the above proposition, the element 1 in Example 4.1.3 is not compact.

In general, we have the following theorem:

Theorem 4.1.7. *Let $(P, \leq)$ be a dcpo.Then,*

$K_P = P$ if and only if $\Downarrow x = \downarrow x$.

Proof. ($\Rightarrow$) Suppose that for each $x \in P$, x is compact. It is clear that $\Downarrow x \subseteq \downarrow x$. Now, let $y \in \downarrow x$ and let D be any directed subset of P such that $x \leq \bigvee D$. Since $y \in \downarrow x$, then $y \leq x$. Thus, $y \leq \bigvee D$. Since y is compact, then there exists $d \in D$ such that $y \leq d$. Hence, $y \ll x$.
($\Leftarrow$) Suppose that $\Downarrow x = \downarrow x$. Then, for each $x, y \in P, x \ll y$ if and only $x \leq y$. Thus, $\forall\ x \in P, x \ll x$ and hence, x is compact. $\square$

Corollary 4.1.8. *In finite posets, each element is compact.*

Proof. Let P be a finite poset. Then, for any $x \in P$, $x \leq x$. Now, from Lemma 4.1.5, $x \ll x$ and hence by Proposition 4.1.6, x is compact. $\square$

Corollary 4.1.9. *In any dcpo, the approximation relation $\ll$ over any subset of compact elements forms a partial order. In particular, over finite posets, the approximation relation defines a partial order.*

Proof. Let P be a dcpo and let A be a subset of compact elements in P. Now, for any $a, b, c \in A$:
(i) $a \ll a$ (by Proposition 4.1.6). That is; $\ll$ is reflexive.
(ii) If $a \ll b$ and $b \ll a$, then $a = b$ (by part (1) of Proposition 4.1.4). That is; $\ll$ is anti-symmetric.
(iii) If $a \ll b$ and $b \ll c$, then $a \ll c$ (by part (2) of Proposition 4.1.4). That is; $\ll$ is transitive. This completes the proof of the first statement. The second statement is straightforward from the above corollary. $\square$

For an application of the approximation relation over ideals (see Definition 1.2.20), we finish this section by the following proposition:

Proposition 4.1.10. *[7] Let P be a poset. Then the following two statements are equivalent:*
(1) $x \ll y$.
(2) For each ideal I of P with supremum, the relation $y \leq \sup I$ implies $x \in I$.

Proof. (1)$\Rightarrow$(2) Suppose that $x \ll y$ and let I be an ideal subset of P such that $\sup I$ exists and $y \leq \sup I$. Since I is directed and $x \ll y$, then there is $i \in I$ such that $x \leq i$. Since I is a lower set, then $x \in I$.

(2)$\Rightarrow$(1) Let U be any directed subset with supremum such that $y \leq \bigvee U$. Consider $I = \downarrow U$. So, I is a directed down set; i.e., I is an ideal. To see this, notice first that I is a down set. Now, let $a, b \in I$. Then there exist $a_1, b_1 \in U$ such that $a \leq a_1$ and $b \leq b_1$. Since U is directed, then there exists $w \in U \subseteq I$ such that $a_1 \leq w$ and $b_1 \leq w$. Therefore, I is directed. Moreover, $\bigvee I$ exists and $\bigvee I = \bigvee U$. This is because for any $a \in I$, there exists $a_1 \in U$ such that $a \leq a_1 \leq \bigvee U$. So, $\bigvee U$ is an upper bound of I. Consider v is any other upper bound of I. So, v is an upper bound of U. Therefore, $\bigvee U \leq v$ and hence $\bigvee U = \bigvee I$. So, $y \leq \bigvee I$. Since (2) holds, then $x \in I$. Therefore, there exists $u \in U$ such that $x \leq u$. Hence, $x \ll y$. $\square$

4.2 Continuous posets

One of the most important applications of the definition of the approximation relation is the definition of the continuous poset that plays a big role in domain theory.

Definition 4.2.1. [9] A poset P is said to be *continuous* if every element in P is the directed supremum of elements that approximate it. That is;, a poset P is said to be continuous if for every $x \in P$ there is a directed subset $D_x \subseteq \Downarrow x$ such that $x = \bigvee D_x$.

Remark 4.2.2. [7] We can introduce an equivalent definition for P to be continuous as follows: A poset P is said to be *continuous* if for all $x \in P$, the set $\Downarrow x = \{y : y \ll x\}$ is directed and has x as its join.

Proof. It is clear that the second definition implies the first. Conversely, suppose $y \ll x$ and $z \ll x$ (i., e., $y, z \in \Downarrow x$). Let $x = \bigvee D_x$ where D_x is directed and $d \ll x$ for each $d \in D_x$. By definition of $\ll$, $y \leq d_1$ and $z \leq d_2$ for some $d_1, d_2 \in D_x$.

Suppose d_3 is larger than both d_1 and d_2 in D_x. Then $y \leq d_3$, $z \leq d_3$ and $d_3 \ll x$. Hence the set $\Downarrow x$ is directed.

Since $y \leq x$ for each $y \in \Downarrow x$, then x is an upper bound of $\Downarrow x$. Suppose that v is any other upper bound of $\Downarrow x$, then v is an upper bound of D_x. Therefore, $\bigvee D_x = x \leq v$. Thus, x is the least upper bound of $\Downarrow x$. Hence, $x = \bigvee \Downarrow x$. □

Example 4.2.1. *The set of real numbers $\mathbb{R}$ under the usual order is a continuous poset.*

Proof. Let $x \in \mathbb{R}$ be arbitrary fixed point and let $z \in \mathbb{R}$. Then we have three cases:

Case 1: $z < x$. Let U be any directed subset with join $\bigvee U$ such that $x \leq \bigvee U$. Then, $z < \bigvee U$ and hence z is not an upper bound of U in $\mathbb{R}$ under usual order. Therefore, there is $u \in U$ such that $z \leq u$ and hence $z \ll x$.

Case 2: $z = x$. Then, $z \not\ll x$. For if we take $U = (-\infty, x)$ which is directed, then $\bigvee U = x$. So, $x \leq \bigvee U$ but there is no $u \in U$ such that $z = x \leq u$.

Case 3: $z > x$. Then, by the contrapositive of part (1) in Proposition 4.1.3, we have $z \not\ll x$.

Thus, from the three cases we have $\Downarrow x = (-\infty, x)$ which is a directed subset with join x. Hence, $\mathbb{R}$ is continuous. □

Here are two examples of non-continuous posets:

Example 4.2.2. *Recall Example 1.2.9.*

Claim: $\Downarrow 1 = \phi$. To see this, assume that there is $x \in D$ such that $x \ll 1$. Then, $x \leq 1$ (otherwise $x \not\ll 1$) and hence $x = 1$. Therefore, $1 \ll 1$. Now, let $S = (2, 3)$. Therefore, S is a directed subset with $\bigvee S = 3$ and that $1 \leq \bigvee S$. Since there is no $s \in S$ such that $1 \leq s$, then $1 \not\ll 1$, which is a contradiction. So, $\Downarrow 1 = \phi$. Thus, for the element $1 \in D, \Downarrow 1$ is not directed and hence D is not continuous.

Example 4.2.3. *[10] A constructive example of a non-continuous poset is obtained by adding a top element ∞ to the natural numbers under their natural order and* *adding an element a with $0 < a \leq \infty$ where a is incomparable with all other elements. Now, let $P = \mathbb{N} \cup \{\infty\} \cup \{a\} \cup \{0\}$. Then, $\Downarrow a = \{0\}$. Thus, $\bigvee \Downarrow a \neq a$. Hence, P is not a continuous poset.*

Recalling the definition of the algebraic dcpo (see Definition 1.2.11), we have the following result:

Proposition 4.2.3. *Every algebraic dcpo is continuous.*

Proof. Let D be an algebraic dcpo. Then, for each $x \in D$ we have that the set $\downarrow_K x = \{a \in K_D : a \leq x\}$ is directed and $x = \bigvee \downarrow_K x$. Only we have to show that$\downarrow_K x \subseteq \Downarrow x$. So, let $a \in \downarrow_K x$. Then, a is compact and $a \leq x$. So, by Proposition 4.1.6, we have $a \leq a \ll a \leq x$ and hence by Proposition 4.1.3 part (2), $a \ll x$. Hence $\downarrow_K x \subseteq \Downarrow x$. By Definition 4.2.1, D is continuous. □

Corollary 4.2.4. *Every finite linearly ordered set is continuous.*

Proof. Let P be a finite linearly ordered set. Then, by Lemma 1.2.14, P is an algebraic dcpo. By Proposition 4.2.3, P is continuous. □

Theorem 4.2.5. *[10] Any finite partially ordered set is continuous, with the approximation relation ($\ll$) coinciding with the partial order.*

Proof. Let $(P, \leq)$ be a finite poset. Then, by Lemma 4.1.5, the approximation relation($\ll$) coincides with the partial order relation($\leq$). Thus, for any $x \in P$, the set

$$\Downarrow x = \downarrow x = \{y : y \leq x\}$$

is directed with x as its join (see Lemma 1.2.3). Hence, P is continuous. □

Example 4.2.4. *[10] The plain is a continuous poset under its coordinatewise order (i.e, for $x, y \in \mathbb{R}^2$, $x \leq y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$), with approximation relation given by $x \ll y$ if and only if $x_1 < y_1$ and $x_2 < y_2$.*

We close this section by the following lemma:

Lemma 4.2.6. *[10] In a continuous poset P, the set $\Downarrow x$ is an ideal for each $x \in P$.*

Proof. Let P be a poset. Then, for any $x \in P$, the set $\Downarrow x$ is directed and has x as its join. Let $y \in \Downarrow x$ and let z be any element of P such that $z \leq y$. So, $z \leq y$ and $y \ll x$. By Proposition 4.1.3 we have $z \ll x$ and hence $z \in \Downarrow x$. Thus, $\Downarrow x$ is a down set. □

4.3 Scott bases

We saw in Proposition 3.2.5 that the family $\uparrow K_D = \{\uparrow a : a \in K_D\}$ is a base for the Scott topology $\sigma(D)$ on an algebraic dcpo D. In this section, we'll give more than one base for the Scott topology over a poset in the shadow of the approximation relation.

We begin first by the definition of the base for a poset.

Definition 4.3.1. [9] Let P be a poset and $B \subseteq P$. B is called a *basis* for P if $\forall\ x \in P$, there is a directed set $D_x \subseteq B$ such that $\forall\ d \in D_x$, $d \ll x$ (i.e., $D_x \subseteq B \cap \Downarrow x$) and $\sup D_x = x$.
In other words, a subset $B \subseteq P$ is a basis for P if each element in P is the directed supremum of elements in B that approximate it.

This together with the definition of a continuous poset lead us to the following lemma:

Lemma 4.3.2. *[8] A poset is continuous if and only if it has a basis.*

Proof. ($\Rightarrow$) Let$P$ be a continuous poset. Then, for each $x \in P$, $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$. In the above definition put $B = P$ and $D_x = \Downarrow x$ to get the result.
($\Leftarrow$) Let P be a poset with $B \subseteq P$ as its basis. So, $\forall\ x \in P$, $\exists D_x \subseteq B$ such that D_x

is directed with $d \ll x \forall\ d \in D_x$ and $x = \bigvee D_x$. That is, x is a directed supremum of elements that approximate it. Hence, P is continuous. $\square$

Example 4.3.1. *The rational points,$\mathbb{Q}$, forms a basis for $\mathbb{R}$ under usual order.*

Proof. Let $x \in \mathbb{R}$ be arbitrary. Then, $\Downarrow x = (-\infty, x)$ (see Example 4.2.1). Let $B = \mathbb{Q} \cap \Downarrow x$. Thus, $B = \{q \in \mathbb{Q} : q < x\}$. It is clear that $x = \bigvee B$. Also, B is a subset of $\mathbb{Q}$, which is linearly ordered. So B is directed (see Lemma 1.2.2). $\square$

Lemma 4.3.3. *[10] The set P of all elements of a continuous poset is a basis for P.*

Proof. Let P be a continuous poset. Then, for any $x \in P$ there is a directed subset $D_x \subseteq \Downarrow x = P \cap \Downarrow x$ such that $x = \bigvee D_x$. Hence, P is a basis for P. $\square$

Theorem 4.3.4. *In a continuous finite poset P, no proper subset is a basis for P.*

Proof. Let P be a continuous finite poset and suppose B is a basis for P and let $y \in P$. Thus, $\Downarrow y \cap B$ contains a directed set D_y such that $y = \bigvee D_y$. Since, in finite posets, each directed subset contains its supremum, therefore $y \in D_y \subseteq \Downarrow y \cap B$ which implies that $y \in B$, and hence $B = P$. $\square$

An important property of the approximation relation on continuous posets is the interpolation property. Before introducing this property, we have the following lemma which is valid in the remainder of this chapter.

Lemma 4.3.5. *[7] Let P be a continuous poset and let $x, y \in P$ such that $x \not\leq y$. Then, there exists $z \in \Downarrow x$ such that $z \not\leq y$.*

Proof. Let $x, y \in P$ such that $x \not\leq y$ and suppose to contrary that for each $z \in \Downarrow x$, $z \leq y$. This implies that y is an upper bound of $\Downarrow x$. Since P is continuous, then $x = \bigvee \Downarrow x$. Therefore, $x \leq y$ which is a contradiction. Hence, we are done. $\square$

Proposition 4.3.6. *[7] If P is a continuous dcpo, then the approximation relation $\ll$ has the interpolation property:* $x \ll z \Rightarrow \exists\, y \in P$ *such that* $x \ll y \ll z$.

Proof. Let $D = \{d \in P : \exists\, b \in P \text{ such that } d \ll b \ll z\}$.

Claim 1: D is directed.

Let $d_1, d_2 \in D$. Then there is $b_1, b_2 \in P$ such that $d_1 \ll b_1 \ll z$ and $d_2 \ll b_2 \ll z$. Since $\Downarrow z$ is directed and $b_1, b_2 \in \Downarrow z$ then, there is $b \in \Downarrow z$, i.,e., $b \ll z$, such that $b_1 \leq b$ and $b_2 \leq b$. So, $d_1 \ll b_1 \leq b \leq z$ and $d_2 \ll b_2 \leq b \leq z$ and hence from Proposition 4.1.3, $d_1 \ll b$ and $d_2 \ll b$. That is; d_1 and $d_2 \in \Downarrow b$. Since $\Downarrow b$ is directed, then there is $d \in \Downarrow b$ such that $d_1 \leq d$ and $d_2 \leq d$. Since $d \ll b \ll z$, then $d \in D$. Hence, D is directed.

Claim 2: D has z as its join.

By the definition of D, $\forall\, d \in D,\ d \leq z$. So, z is an upper bound of D. Let $p \in P$ such that $p = \bigvee D$. Then, $p \leq z$. Suppose that $p < z$, i.e., $z \nleq p$. Since P is continuous, then $z = \sup \Downarrow z$. Therefore, by Lemma 4.3.5, there exists $t \ll z$ such that $t \nleq p$. Similarly, there exists $s \in \Downarrow t$ such that $s \nleq p$. Thus, $s \ll t \ll z$ for some $t \in P$ implies that $s \in D$. However, $s \in D$, $p = \sup D$ and $s \nleq p$ is a contradiction. Hence $z \leq p$. Thus, $z = \sup D$.

Now, given $x \ll z$, since D is directed and $z = \sup D$, then there exists $d \in D$ such that $x \leq d$. So, by the definition of D, there exists $b \in P$ such that $x \leq d \ll b \ll z$. Hence, again by Proposition 4.1.3, $x \ll b \ll z$ and hence the interpolation property holds. □

Corollary 4.3.7. *[7] Let P be a continuous dcpo. If $x \ll y$ and $y \leq \sup D$, for a directed subset D of P, then $x \ll d$ for some $d \in D$. (Note that the desired result is $x \ll d$ not $x \leq d$ as in the definition of the approximation relation.)*

Proof. Suppose that $x \ll y$ and $y \leq \sup D$ for a directed set D. By the interpolation property, there exists $w \in P$ such that $x \ll w \ll y$. Since $w \ll y$, there exists $d \in D$ such that $w \leq d$. Thus, by Proposition 4.1.3 part(2), $x \ll d$. □

The interpolation property plays an important role in the theory of continuous posets; it states that if $x \ll y$, then we can interpolate an additional element between them (note that this element might be x or y when $x \ll x$ or $y \ll y$). Corollary 4.3.7 can be viewed as an alternate version of the interpolation property.

One important application of the interpolation property is stated in the next proposition that associates the approximation relation with the Scott topology. Before introducing this proposition we pave by the following:

Definition 4.3.8. [7] A filter F in a poset P is called an *open filter* if it is a Scott open set. An open filter F is called *locally bounded* if for all $y \in F$, there exists an open filter $G_y \subseteq F$ and there exists $z \in F$ such that $y \in G_y$ and $G_y \subseteq \uparrow z$.

Lemma 4.3.9. *Let X be a poset. Then, for any $x, y \in X$, $y \in$int$(\uparrow x)$ (where int$(\uparrow x)$ denotes the interior of $\uparrow x$ in the Scott topology) implies $x \ll y$.*

Proof. Let D be any directed subset of X such that $\sup D$ exists and $y \leq \sup D$. Since $y \in$ int$(\uparrow x)$, $y \leq \sup D$ and since int$(\uparrow x)$ is an up set, then $\sup D \in$ int$(\uparrow x)$. Since int$(\uparrow x)$ is Scott-open, there exists $d \in D$ such that $d \in$ int$(\uparrow x)$ $\subseteq \uparrow x$. So, $d \in \uparrow x$ and $x \leq d$. Hence, $x \ll y$. □

Proposition 4.3.10. *[7] Let P be a continuous dcpo and let $x, y \in P$. Then the following are equivalent:*

(1) $x \ll y$

(2) There exists an open filter G with $y \in G \subseteq \uparrow x$.

(3) $y \in$ int$(\uparrow x)$ (where int$(\uparrow x)$ denotes the interior of $\uparrow x$ in the Scott topology).

Proof. (1)($\Rightarrow$)(2). Since $x \ll y$ and P is continuous, then by the interpolation property, there exists x_1 such that $x \ll x_1 \ll y$. Also, there exists x_2 such that $x \ll x_2 \ll x_1$. Thus, continuing in this process we can obtain $\{x_n\}_{n \in \mathbb{N}}$ satisfying $x \ll x_{n+1} \ll x_n \ll y$ for all n. Let $G = \cup_n(\uparrow x_n)$.

Claim: G is Scott open.

Clearly, G is an up set. Let D be any directed subset of P such that $\sup D \in G$. Then $x_n \leq \sup D$ for some n. Since $x_{n+1} \ll x_n$, then by the definition of $\ll$, $x_{n+1} \leq d$ for some $d \in D$, and thus $d \in \uparrow x_{n+1} \subseteq G$. Hence, G is Scott open.

Now, from Proposition 4.1.3, we have $x \leq x_{n+1} \leq x_n \leq y$ for each n. Thus, G is a filtered set and $y \in G \subseteq \uparrow x$.

(2)($\Rightarrow$)(3) Immediate (since G is Scott open).

(3)($\Rightarrow$) (1) The proof follows immediately by applying Lemma 4.3.9. $\square$

Corollary 4.3.11. *[7] Let P be a continuous dcpo and let $x \in P$. Then, $\Uparrow x =$int($\uparrow x$), where the interior of $\uparrow x$ is taken with respect to the Scott topology. Hence, $\Uparrow x$ is Scott open.*

Proof. According to the above proposition, we can say that, in any continuous dcpo P, $y \in \Uparrow x$ if and only if $y \in \mathrm{int}(\uparrow x)$, for $x, y \in P$. Therefore $\Uparrow x = \mathrm{int}(\uparrow x)$ and hence is a Scott open. $\square$

Proposition 4.3.12. *[7] Let P be a continuous dcpo. Then*

(i) $\{\Uparrow x : \ x \in P\}$ is a base for the Scott topology, and

(ii) The set of all filters is a base for the Scott topology.

Proof. (i) Let U be a Scott open set and let $y \in U$. Since P is continuous, then $\Downarrow y$ is a directed set and $y = \sup \Downarrow y$. Also, $U \cap \Downarrow y \neq \phi$. Thus, there exists $x \in U$ such that $x \ll y$, i.e.; $y \in \Uparrow x \subseteq U$. By Corollary 4.3.11, we have $\Uparrow x = \mathrm{int}(\uparrow x) \in \sigma(P)$. Also, $y \in \Uparrow x \subseteq \uparrow U = U$.

(ii) Let U be a Scott open set and let $y \in U$. Since P is continuous, then $\Downarrow y$ is directed and $y = \bigvee \Downarrow y$. So, $U \cap \Downarrow y \neq \phi$. So, there exists $x \in U$ such that $x \ll y$. By Proposition 4.3.10, there exists an open filter F such that $y \in F \subseteq \uparrow x \subseteq U$. Hence, the collection of open filters forms a base for the Scott topology. $\square$

Here we have the following Proposition:

Proposition 4.3.13. *[7] Let P be a dcpo. The following statements are equivalent:*
(1) P is continuous.
(2) Every open filter is locally bounded, the open filters containing a fixed point form a descending family and the open filters separate the points of P.
(3) For each $x \in P$, $\{y \in P : x \in int(\uparrow y)\}$ is directed and has x as its supremum.

Proof. (1)⇒(2) Let F be an open filter and $y \in F$. Since $y \in P$ and P is continuous, then $\Downarrow y$ is directed and $y = \sup \Downarrow y$. Since F is Scott open with $y \in F$, then $F \cap \Downarrow y \neq \phi$. So, there exists $x \ll y$ with $x \in F$. By Proposition 4.3.10, there exists an open filter G with $y \in G \subseteq \uparrow x \subseteq \uparrow F = F$. Thus, F is locally bounded. Since the open filters are a base for the Scott topology (Proposition 4.3.12), they form a descending family at each point.
Now, suppose $y \neq z$, for some $z \in P$. Without loss of generality, we may assume that $y \not\leq z$. Then, by Lemma 4.3.5, there exists $x \ll y$ such that $x \not\leq z$ and hence $z \notin \uparrow x$. By Proposition 4.3.10, there exists an open filter G such that $y \in G \subseteq \uparrow x$. Hence, $z \notin G$.
(2)⇒(1) Let $x \in P$ and $w < x$, for some $w \in P$. Since open filters separate points, then there exists an open filter F such that $x \in F$ and $w \notin F$. Since F is locally bounded, there exists an open filter $G \subseteq F$ and $z \in F$ such that $x \in G \subseteq \uparrow z$. Note that $z \not\leq w$ (since F is an up set and $z \in F$ and $w \notin F$). Now, by Proposition 4.3.10, $x \in G \subseteq \mathrm{int}(\uparrow z) \subseteq \uparrow z$ implies $z \ll x$, and hence $z \leq x$. Let H denote the set $\{z \in P :$ there exists an open filter G such that $x \in G \subseteq \uparrow z\}$. Certainly, we have $\sup H \leq x$, if $\bigvee H$ exists. We have just shown that for $w < x$, there exists such a z with $z \not\leq w$. Hence equality (i.e., $x = \sup H$) must hold, provided the sup exists. Since for each $z \in H$ we have $z \ll x$, we complete the proof by showing H is directed(and hence guaranteeing the existence of the sup) and appealing to Definition 4.2.1. Let $F_1,\ F_2$ be open filters, $x \in F_1 \cap F_2$ and $F_1 \subseteq \uparrow z_l, F_2 \subseteq \uparrow z_2$. By hypothesis, there exists an open filter F such that $x \in F \subseteq F_1 \cap F_2$; local

boundedness implies that there exists an open filter $G \subseteq F$ and $z_3 \in F$ such that $x \in G \subseteq \uparrow z_3$. Then $z_1 \leq z_3$ and $z_2 \leq z_3$.

(1)⇒(3) Since P is continuous, then the set $\{y \in P : x \in int(\uparrow y)\} = \{y \in P : y \ll x\} = \Downarrow x$. Thus, the required result follows immediately by applying Remark 4.2.2.

(3)⇒(1) Suppose $x \in \mathrm{int}(\uparrow y)$. Let D be any directed subset of P such that $x \leq \sup D$. Then, $\sup D \in \mathrm{int}(\uparrow y)$ since $\mathrm{int}(\uparrow y)$ is Scott open. Thus, $D \cap \mathrm{int}(\uparrow y) \neq \phi$. So, $d \in \mathrm{int}(\uparrow y)$ for some $d \in D$, and hence $y \leq d$. This argument shows $y \ll x$ (see Lemma 4.3.9). It follows that $y \in \Downarrow x$, and therefore $\{y \in P : x \in int(\uparrow y)\} \subseteq \Downarrow x$. Since x is arbitrary and the set $\{y \in P : x \in int(\uparrow y)\}$ is a directed set of elements that approximate x with x as its join, the desired conclusion follows by applying Definition 4.2.1. □

Conclusion

The aim of this research is to focus on the Scott topology and some of its properties and furthermore, its relation with the T_0-Alexandroff topology. The research pointed out the relation that the Scott topology is coarser than the T_0-Alexandroff topology and they are the same on finite posets.

The researcher looking for more studies in the future Studying application studies on the Scott topology. Moreover, Looking for a definite definitions of the interior, exterior, boundary and limit points in the Scott topology.

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